On Steklov Eigenproblems

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Vladimir Andreevich Steklov (1864 - 1926)



Steklov was not only an outstanding mathematician, who made many important contributions to Applied Mathematics, but also had an unusually bright personality.
The Mathematical Institute of the Russian Academy of Sciences in Moscow bears his name.
On his life and work, see Kuznetsov, Kulczycki, Kwasnicki, Nazarov, Poborchi, Polterovich & Siudeja. "*The Legacy of Vladimir Andreevich Steklov*" (Notices of the AMS, 2014).

- Boundary spectral parameters were introduced by Steklov, in 1902.

The classical Steklov eigenproblem

The classical Steklov eigenproblem reads

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u, & \text{on } \Gamma := \partial\Omega, \end{cases} \tag{S}$$

where ν denotes the unit outer normal to Γ, and the unknown u is a (real-, or complex-valued) function called Steklov eigenfunction, while the unknown number λ is called Steklov eigenvalue.

Ω is a sufficiently regular[1] bounded domain[2] in $\mathbb{R}^n$, and the (scalar) harmonic function u is required to belong to the standard Sobolev space $H^1(\Omega)$.

(S) can be considered as the eigenvalue problem for the celebrated Dirichlet-to-Neumann map:

[1] Γ is at least Lipschitz continuous.
[2] Domain = Connected open set.

Dirichlet-to-Neumann map

Given the solution $u \in H^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma, \end{cases} \tag{D}$$

with datum[3] $f \in H^{1/2}(\Gamma)$, one can consider the normal derivative $\frac{\partial u}{\partial \nu}$ of u as an element of $H^{-1/2}(\Gamma)$.

This allows to define the map $\mathcal{D}$ from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ by setting

$$\mathcal{D}f = \frac{\partial u}{\partial \nu}\,.$$

$\mathcal{D}$ is called the Dirichlet-to-Neumann map, and its eigenpairs (f, λ) correspond to the eigenpairs (u, λ) of problem (S), f being the trace of u on Γ.

[3] For a "nice" Ω, there exists a unique linear continuous map (the trace of u on $\partial\Omega$) from $H^1(\Omega)$ into $L^2(\partial\Omega)$, such that for any $u \in H^1(\Omega) \cap C(\overline{\Omega})$ we have $\gamma_0(u) = \mathrm{Tr}(u) := u|_{\partial\Omega}$. Then $H^{1/2}(\Omega) := \gamma_0(H^1(\Omega))$, and $H^{-1/2}(\Omega)$ is its dual.

Applications

There is a broad spectrum of applications of the Steklov eigenvalue problem in various areas, including

- Calderón's problem (the inverse problem of recovering the electric conductivity of an electric body from the knowledge of the voltage-to-current map).
- Spectral geometry.
- Shape optimisation.
- General 2nd order elliptic operators.
- Differential forms on a compact Riemannian manifold with smooth boundary.
- The biharmonic equation (4th order) - Elasticity.
- Travelling waves for nonlinear pdes (e.g., the defocusing NLS).
- Maximising the information transmission rate in the cerebral cortex.

Nevertheless, time allows only a couple of words about:

Applications - Linear water waves

Consider water waves in a canal, open sea, or other unbounded domain.
Linear water waves are described by a mixed BVP for the Laplace equation with the Steklov spectral boundary condition on the horizontal water surface.
Wave propagation occurs provided the Steklov spectral parameter λ belongs to the continuous spectrum σ_c of the problem.
In an infinite basin Ω, this spectrum is not empty and usually includes the positive real axis of $\mathbb{C}$. The inclusion $\lambda \in \sigma_c$ frustrates the Fredholmness of the operator in $H^2(\Omega)$, and one needs to reduce the data space and to impose radiation conditions (which distinguish between the waves incoming from, or outgoing to, infinity).
Nazarov & Taskinen (2010) have shown that σ_c may be nonempty even in a bounded 3-dim pond.
This is due to either a submerged body touching the water surface, or a sharp edge of the pond.

Kuznetsov, Maz'ya & Vainberg, *Linear Water Waves - A Mathematical Approach*, 2004.

The sloshing problem consists in the study of small oscillations of a liquid in a finite basin represented by a bounded domain Ω in $\mathbb{R}^3$, with $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 corresponds to the horizontal (free) surface of the liquid at rest, and Γ_2 to the bottom of the basin.
The Steklov boundary condition $\frac{\partial u}{\partial \nu} = \lambda u$ is imposed only on Γ_1, while the Neumann condition $\frac{\partial u}{\partial \nu}$ is imposed on Γ_2.
Canavati & Minzoni (1979) studied existence and regularity for such mixed BVPs. They proved stronger regularity results (than the general ones of Miranda (1955)), under very special assumptions on the boundary and on the data. They reformulated the Steklov problem in terms of a nonlocal operator, and established the existence of a purely point spectrum for the inverse operator. These problems are very similar to regular Sturm-Liouville problems. Nevertheless, there are situations of interest (e.g., waves on sloping beaches), that produce singular Steklov problems (the boundary can have cones, or be infinite). Special cases of this situation were studied by Whitham & Minzoni (1977), in connection with wave propagation.

Mayer & Krechetnikov (2012): dynamics of liquid sloshing

The velocity potential $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} \Delta\Phi(x,t) = 0, \text{ inside the mug,} \\ \dfrac{\partial^2\Phi(x,t)}{\partial t^2} + g\,\dfrac{\partial\Phi(x,t)}{\partial\nu} = 0, \text{ on the free surface,} \\ \dfrac{\partial\Phi(x,t)}{\partial\nu} = 0, \text{ on the sides and the bottom.} \end{cases}$$

Separation of variables leads to the following problem for the x-component (f) of Φ

$$\begin{cases} \Delta f = 0, \text{ inside the mug,} \\ \dfrac{\partial f}{\partial\nu} = \dfrac{\lambda}{g} f, \text{ on the free surface,} \\ \dfrac{\partial f}{\partial\nu} = 0, \text{ on the bottom.} \end{cases}$$

Time-harmonic Maxwell's equations

When one considers time-harmonic fields ($\boldsymbol{U}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$, where $\omega > 0$), in a *linear homogeneous isotropic* electromagnetic medium (i.e., with constant physical parameters) in $\mathbb{R}^3$, the *time-harmonic Maxwell's equations*

$$\operatorname{curl} \mathbf{E} - \mathrm{i}\,\omega\mu\, \mathbf{H} = \mathbf{0}, \operatorname{curl} \mathbf{H} + \mathrm{i}\,\omega\varepsilon\, \mathbf{E} = \mathbf{0},$$

where ω is the angular frequency, $\varepsilon \in \mathbb{R}$ and $\mu \in \mathbb{R}$ are the *electric permittivity*, and the *magnetic permeability* of the medium, and $\boldsymbol{E}, \boldsymbol{H}$ are the (space-dependent parts of) the electric and the magnetic field, respectively, are satisfied.

Since both ε and μ are assumed to be constant, we have that $\mathbf{E}$ and $\mathbf{H}$ are automatically divergence-free:

$$\operatorname{div} \mathbf{E} = \operatorname{div} \mathbf{H} = 0.$$

Table: **Electromagnetic Metamaterials**

$\varepsilon > 0$ & $\mu > 0$: dielectrics	$\varepsilon > 0$ & $\mu < 0$: gyrotropic
$\varepsilon < 0$ & $\mu > 0$: plasmas	$\varepsilon < 0$ & $\mu < 0$: left-handed

The interior Calderón operator

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary Γ.
We consider the following classical BVP that involves the perfect conductor condition on Γ

$$\begin{cases} \text{curl}\,\mathbf{E} - i\omega\mu\mathbf{H} = \mathbf{0}\,,\ \ \text{curl}\,\mathbf{H} + i\omega\varepsilon\mathbf{E} = \mathbf{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \times \mathbf{E} = \mathbf{m}, & \text{on } \Gamma, \end{cases} \qquad ((\mathbf{E},\mathbf{H})\text{-PEC})$$

where $\boldsymbol{\nu}$ denotes the unit outer normal to Γ.

The interior Calderón operator is defined as the mapping of the tangential component of the electric field to the tangential component of the magnetic field on Γ, i.e., $\mathbf{m} \mapsto \boldsymbol{\nu} \times \mathbf{H}$.

Calderón operators are also called Poincaré-Steklov, or impedance, or admittance, or capacity operators.

By eliminating $\mathbf{H}$ we obtain

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = \mathbf{0}, & \text{in } \Omega, \\ \boldsymbol{\nu}\times\mathbf{E} = \mathbf{m}, & \text{on } \Gamma, \end{cases} \tag{$\mathbf{E}$-PEC}$$

where $k^2 := \omega^2\varepsilon\mu$.

Instead of the standard interior Calderón operator for ($\mathbf{E}$-PEC) (see, e.g., Cessenat (1996), Kristensson, $\mathbb{S}$, Wellander & Yannacopoulos (2020)), we consider its variant defined by $\mathbf{m}\mapsto(\boldsymbol{\nu}\times\mathbf{H})\times\boldsymbol{\nu}$, i.e.,

$$\boldsymbol{\nu}\times\mathbf{E}\mapsto -\frac{\mathrm{i}}{\omega\mu}(\boldsymbol{\nu}\times\operatorname{curl}\mathbf{E})\times\boldsymbol{\nu}\,.$$

The natural analogue in Electromagnetics (so in $\mathbb{R}^3$) of the classical Steklov problem (S), can be defined as the eigenvalue problem for the rescaled (of the defined in the previous slide) interior Calderón operator

$$\boldsymbol{\nu} \times \mathbf{E} \mapsto -(\boldsymbol{\nu} \times \operatorname{curl} \mathbf{E}) \times \boldsymbol{\nu}.$$

Therefore, one looks for values λ such that $(\boldsymbol{\nu} \times \operatorname{curl} \mathbf{E}) \times \boldsymbol{\nu} = -\lambda \boldsymbol{\nu} \times \mathbf{E}$, or, equivalently (by taking another cross product by $\boldsymbol{\nu}$)

$$\boldsymbol{\nu} \times \operatorname{curl} \mathbf{E} = \lambda \mathbf{E}_{\mathrm{T}},$$

where $\mathbf{E}$ satisfies $\operatorname{curl} \operatorname{curl} \mathbf{E} - k^2 \mathbf{E} = \mathbf{0}$, and $\mathbf{E}_{\mathrm{T}} := (\boldsymbol{\nu} \times \mathbf{E}) \times \boldsymbol{\nu}$ is the tangential component[4] of $\mathbf{E}$.

The Steklov eigenvalue problem for Maxwell's equations is

$$\begin{cases} \operatorname{curl} \operatorname{curl} \mathbf{E} - k^2 \mathbf{E} = \mathbf{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \times \operatorname{curl} \mathbf{E} = \lambda \mathbf{E}_{\mathrm{T}}, & \text{on } \Gamma. \end{cases} \qquad (SM)$$

[4] For $\mathbf{U} : \overline{\Omega} \to \mathbb{R}^3$, we have $\mathbf{U}|_\Gamma = (\boldsymbol{\nu} \cdot \mathbf{U}|_\Gamma)\, \boldsymbol{\nu} + (\boldsymbol{\nu} \times \mathbf{U}|_\Gamma) \times \boldsymbol{\nu}$.

(SM) was first introduced, for $k > 0$, by Camanõ, Lackner & Monk (2017), where it was pointed out that the spectrum is not discrete. In particular, for the case of the unit ball in $\mathbb{R}^3$, it turns out that the eigenvalues consist of two infinite sequences, one of which is divergent and the other is converging to zero. To overcome this issue, they considered a modified problem having discrete spectrum and then used it for the study of an inverse scattering problem.

On the other hand, Lamberti & $\mathbb{S}$ (2020) have analysed (SM), for $k^2 \in \mathbb{R}$, only for tangential vector fields $\mathbf{E}$, in which case (SM) can be written as

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = \mathbf{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \times \operatorname{curl}\mathbf{E} = \lambda\mathbf{E}, & \text{on } \Gamma. \end{cases} \qquad (SM\text{-}tan)$$

The boundary condition automatically implies that $\mathbf{E}$ is tangential, hence the null sequence of eigenvalues disappears and the spectrum turns out to be discrete.

There are not so many publications devoted to Steklov problems for Maxwell's equations: apart the two papers mentioned above, we know of Cakoni, Cogar & Monk (2021), Cogar (2020), (2021), (2022), Cogar, Colton & Monk (2019), Cogar & Monk (2020), and Halla (2019), (2021).

The main problem

Ω denotes a bounded domain in $\mathbb{R}^3$ with a $C^{1,1}$ boundary[5] $\Gamma := \partial\Omega$.

The energy space is $X_{\mathrm{T}}(\Omega) = \{\boldsymbol{U} \in (H^1(\Omega))^3 : \ \boldsymbol{U} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma\}$.

To guarantee the coercivity of the quadratic form associated with the corresponding differential operator, we employ an idea of Costabel & Dauge (1999), by introducing a penalty term $\theta\, \mathrm{grad}\, \mathrm{div}\, u$ in the equation, where θ can be any *positive* number.

Namely, we consider the eigenvalue problem[6]

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E} - k^2\boldsymbol{E} - \theta\ \mathrm{grad}\ \mathrm{div}\,\boldsymbol{E} = \boldsymbol{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \times \mathrm{curl}\,\boldsymbol{E} = \lambda\boldsymbol{E}, & \text{on } \Gamma, \\ \boldsymbol{E} \cdot \boldsymbol{\nu} = 0, & \text{on } \Gamma, \end{cases} \qquad (SM\text{-}\theta)$$

where $\boldsymbol{E}$ is the unknown vector field.

- We do not assume that $k^2 := \omega^2\varepsilon\mu \in \mathbb{R}$ is necessarily positive.

Hence, all combinations of signs are allowed for ε and μ (*electromagnetic metamaterials*, recall slide 10).

[5] A $C^{1,1}$ boundary is the graph of an everywhere differentiable function, having a locally Lipschitz continuous gradient.

[6] Although the second boundary condition is in fact embodied in the first one, we still include it in $(SM\text{-}\theta)$ since it appears in the definition of the energy space.

Weak formulation

(SM-θ) has to be interpreted in the weak sense as: find $\boldsymbol{E} \in X_{\mathrm{T}}(\Omega)$ such that

$$\int_\Omega \operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \varphi \, dx - k^2 \int_\Omega \boldsymbol{E} \cdot \varphi \, dx + \theta \int_\Omega \operatorname{div} \boldsymbol{E} \operatorname{div} \varphi \, dx = -\lambda \int_\Gamma \boldsymbol{E} \cdot \varphi \, d\sigma \,, \qquad (wSM\text{-}\theta)$$

for all $\varphi \in X_{\mathrm{T}}(\Omega)$.

We need to **assume that** k^2 **does not coincide with any eigenvalue** A_n **of**

$$\begin{cases} \operatorname{curl} \operatorname{curl} \boldsymbol{E} - \theta \operatorname{grad} \operatorname{div} \boldsymbol{E} = A\boldsymbol{E} & \text{in } \Omega, \\ \boldsymbol{\nu} \times \boldsymbol{E} = \mathbf{0} & \text{on } \Gamma, \\ \boldsymbol{E} \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma. \end{cases}$$

Clearly the two boundary conditions above are equivalent to the Dirichlet boundary condition $\boldsymbol{E} = \mathbf{0}$ on Γ. So, equivalently, we consider the problem

$$\begin{cases} \operatorname{curl} \operatorname{curl} \boldsymbol{E} - \theta \operatorname{grad} \operatorname{div} \boldsymbol{E} = A\boldsymbol{E} & \text{in } \Omega, \\ \boldsymbol{E} = \mathbf{0} \text{ on } \Gamma. \end{cases} \qquad (Dir)$$

(Dir) has a discrete spectrum which consists of a sequence $\{A_n\}_{n\in\mathbb{N}}$ of positive eigenvalues of finite multiplicity.
Hence the above **assumption** becomes $k^2 \neq A_n$, for all $n \in \mathbb{N}$.

The eigenvalues

The first eigeinvalue of (*Dir*) is

$$A_1 = \min_{\varphi \in (H_0^1(\Omega))^3 \ \ \varphi \neq 0} \frac{\int_\Omega |\operatorname{curl} \varphi|^2 \, dx + \theta \int_\Omega |\operatorname{div} \varphi|^2 \, dx}{\int_\Omega |\varphi|^2 \, dx} > 0.$$

We consider the case

$$k^2 < A_1.$$

The key result is

Theorem

Let $k^2 < A_1$ and $\theta > 0$. The eigenvalues of (SM-θ) are real, have finite multiplicity and can be represented by a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$, divergent to $-\infty$. Moreover, we have the min-max representation:

$$\lambda_n = - \min_{\substack{V \subset X_{\mathrm{T}}(\Omega) \\ \dim V = n}} \ \max_{\varphi \in V \setminus (H_0^1(\Omega))^3} \frac{\int_\Omega \left(|\operatorname{curl} \varphi|^2 - k^2 |\varphi|^2 + \theta |\operatorname{div} \varphi|^2 \right) dx}{\int_\Gamma |\varphi|^2 \, dx}. \qquad (MinMax)$$

The eigenvalues

By a technical procedure we introduce a compact selfadjoint operator $\mathcal{T}$ from $X_{\mathrm{T}}(\Omega)$ to itself; hence its spectrum consists of zero and a decreasing divergent sequence of positive eigenvalues β_j, defined by $\beta_j = (-\lambda_j + \eta)^{-1}$.

Then the characterisation in $(MinMax)$ follows by the classical *Min-Max Principle* applied to $\mathcal{T}$.

$X_{\mathrm{T}}(\Omega)$ can be decomposed as an orthogonal sum

$$X_{\mathrm{T}}(\Omega) = \mathrm{Ker}\mathcal{T} \oplus (\mathrm{Ker}\mathcal{T})^{\perp} = (H_0^1(\Omega))^3 \oplus \mathcal{H}(\Omega),$$

where

$$\mathcal{H}(\Omega) := (\mathrm{Ker}\mathcal{T})^{\perp} = \Big\{ \boldsymbol{E} \in X_{\mathrm{T}}(\Omega) : \int_\Omega \mathrm{curl}\, \boldsymbol{E} \cdot \mathrm{curl}\, \varphi \, dx - k^2 \int_\Omega \boldsymbol{E} \cdot \varphi \, dx + \theta \int_\Omega \mathrm{div}\, \boldsymbol{E}\, \mathrm{div}\, \varphi \, dx = 0, \ \forall \varphi \in (H_0^1(\Omega))^3 \Big\}. \qquad (\star)$$

Note that $\boldsymbol{E} \in \mathcal{H}(\Omega)$ if and only if $\boldsymbol{E}$ is a weak solution in $(H^1(\Omega))^3$ of

$$\begin{cases} \text{curl curl}\, \boldsymbol{E} - k^2 \boldsymbol{E} - \theta \,\text{grad div}\, \boldsymbol{E} = \mathbf{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \cdot \boldsymbol{E} = 0, & \text{on } \Gamma. \end{cases} \tag{‡}$$

Solutions to (‡) play the same role as that played by the usual scalar harmonic functions for the classical Steklov problem.
Further, the eigenfunctions, associated with the eigenvalues β_n, define a complete orthonormal system of $\mathcal{H}(\Omega)$.

The general case

$$A_n < k^2 < A_{n+1},$$

is more demanding; for the proof, see Lamberti & $\mathbb{S}$ (2020).

When is 0 an eigenvalue of $(wSM\text{-}\theta)$?

Let $\Sigma = \{\lambda_n : n \in \mathbb{N}\}$ (the set of eigenvalues of $(wSM\text{-}\theta)$.
Consider two auxiliary problems; the first is the classical Neumann eigenvalue problem for the Laplacian

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0, & \text{on } \Gamma, \end{cases}$$

which admits a divergent sequence $\lambda_n^{\mathcal{N}}$, $n \in \mathbb{N}$, of non-negative eigenvalues of finite multiplicity, with $\lambda_1^{\mathcal{N}} = 0$.
The second is the eigenproblem

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{\psi} = \lambda\boldsymbol{\psi}, & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{\psi} = 0 & \text{in } \Omega, \\ \boldsymbol{\nu} \times \operatorname{curl}\boldsymbol{\psi} = \mathbf{0}, & \text{on } \Gamma, \\ \boldsymbol{\psi}\cdot\boldsymbol{\nu} = 0, & \text{on } \Gamma, \end{cases}$$

which admits a divergent sequence $\lambda_n^{\mathcal{M}}$, $n \in \mathbb{N}$, of non-negative eigenvalues of finite multiplicity.

Theorem

Let $k \neq 0$ and $\theta > 0$. Then

$$0 \in \Sigma \Longleftrightarrow k^2 \in \{\theta\lambda_n^{\mathcal{N}} : n \in \mathbb{N}\} \cup \{\lambda_n^{\mathcal{M}} : n \in \mathbb{N}\}$$

We denote by $\boldsymbol{E}_n^{\Omega}$, $n \in \mathbb{N}$, an orthonormal sequence of eigenvectors (normalised with respect to $\mathcal{Q}$) associated with the eigenvalues λ_n of $(SM\text{-}\theta)$.

Let π_{T} denote the tangential components trace operator[7] from $X_{\mathrm{T}}(\Omega)$ to $TL^2(\Gamma)$, where

$$TL^2(\Gamma) = \{\boldsymbol{u} \in (L^2(\Gamma))^3 : \boldsymbol{\nu} \cdot \boldsymbol{u} = 0 \text{ on } \Gamma\}.$$

By setting

$$\boldsymbol{E}_n^{\Gamma} := \sqrt{|\lambda_n - \eta|}\, \pi_{\mathrm{T}}\left(\boldsymbol{E}_n^{\Omega}\right),$$

it is proved that $\boldsymbol{E}_n^{\Gamma}$, $n \in \mathbb{N}$, is an orthonormal basis of $TL^2(\Gamma)$.

These bases can be used to represent the solutions of

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{U} - k^2\boldsymbol{U} - \theta \operatorname{grad}\operatorname{div}\boldsymbol{U} = \boldsymbol{0}, & \text{in } \Omega, \\ \boldsymbol{\nu} \times \operatorname{curl}\boldsymbol{U} = \boldsymbol{f}, & \text{on } \Gamma, \end{cases} \qquad (\dagger)$$

where $\boldsymbol{f} \in TL^2(\Gamma)$.

[7] $\pi_{\mathrm{T}} : \mathbf{U} \mapsto (\boldsymbol{\nu} \times \mathbf{U}|_{\Gamma}) \times \boldsymbol{\nu} =: \mathbf{U}_{\mathrm{T}}$.

Trace problems and Steklov expansions

Let $\boldsymbol{f}$ be represented as

$$\boldsymbol{f} = \sum_{n=1}^{\infty} c_n \boldsymbol{E}_n^{\Gamma},$$

with $\{c_n\}_{n\in\mathbb{N}} \in \ell^2$.
It is proved in Lamberti & $\mathbb{S}$ (2020), that if $0 \notin \Sigma$ then the solution $\boldsymbol{U}$ of (†) can be expanded as

$$\boldsymbol{U} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{|\lambda_n - \eta|}}{\lambda_n} \, c_n \right) \boldsymbol{E}_n^{\Omega} .$$

Finally, the trace space of $X_{\mathrm{T}}(\Omega)$ can be represented as

$$\pi_{\mathrm{T}}(X_{\mathrm{T}}(\Omega)) = \pi_{\mathrm{T}}(\mathcal{H}(\Omega)) = \left\{ \sum_{j=1}^{\infty} c_j \boldsymbol{E}_j^{\Gamma} : \ \sum_{j=1}^{\infty} |\lambda_j - \eta| |c_j|^2 < \infty \right\}.$$

Scalar Spherical Harmonics

The scalar spherical harmonics are the angular part of the solution to Laplace's equation in spherical coordinates where azimuthal symmetry is not present:

$$Y_{\sigma m \ell}(\vartheta, \varphi) = \sqrt{\frac{\varepsilon_m}{2\pi}} \sqrt{\frac{(2\ell+1)(\ell-m)!}{2(\ell+m)!}}\, P_\ell^m(\cos\vartheta)\, \Phi_\sigma(\varphi),$$

where

- $\sigma \in \{\mathrm{e}, \mathrm{o}\}$,
- $\ell, m \in \mathbb{N}_0 : m \leq \ell$,
- $\vartheta \in [0, \pi]$,
- $\varphi \in [0, 2\pi)$,
- $\varepsilon_m := 2 - \delta_{m0}$ ("Neumann factor"),
- $P_\ell(x) := \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$ (Legendre polynomial),
- $P_\ell^m(x) := (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$ (Legendre function),
- $\Phi_{\mathrm{e}}(\varphi) := \cos(m\varphi)$,
- $\Phi_{\mathrm{o}}(\varphi) := \sin(m\varphi)$.

The spherical harmonics define a complete basis over the unit sphere in $\mathbb{R}^3$.

Vector Spherical Harmonics

We employ the multi-index notation[8] $Y_n := Y_{\sigma m \ell}$.
Let $\mathbb{J}$ be the set of triple indices $\sigma m \ell$, where $\sigma \in \{\mathrm{e}, \mathrm{o}\}$ and $\ell, m \in \mathbb{N}_0 : m \le \ell$ and B be the unit ball in $\mathbb{R}^3$.
For $n \in \mathbb{J}$, and all $\boldsymbol{\xi} \in \partial B$, the vector spherical harmonics are defined as

$$\boldsymbol{A}_{1n}(\boldsymbol{\xi}) = \frac{1}{\sqrt{\ell(\ell+1)}} \operatorname{grad}_{\boldsymbol{\xi}} Y_n(\boldsymbol{\xi}) \times \boldsymbol{\xi},$$

$$\boldsymbol{A}_{2n}(\boldsymbol{\xi}) = \frac{1}{\sqrt{\ell(\ell+1)}} \operatorname{grad}_{\boldsymbol{\xi}} Y_n(\boldsymbol{\xi}),$$

$$\boldsymbol{A}_{3n}(\boldsymbol{\xi}) = Y_n(\boldsymbol{\xi})\, \boldsymbol{\xi},$$

along with $\boldsymbol{A}_{1\sigma 00} = \boldsymbol{A}_{2\sigma 00} = \boldsymbol{0}$.

- $\{\boldsymbol{A}_{\tau n} : \tau \in \{1,2,3\}, n \in \mathbb{J}\}$ is a complete orthonormal system in $(L^2(\partial B))^3$.

For $\boldsymbol{x} \in B \setminus \{\boldsymbol{0}\}$, we set

$$\boldsymbol{A}_{1n}(\boldsymbol{x}) = \frac{|\boldsymbol{x}|}{\sqrt{\ell(\ell+1)}} \operatorname{grad}_{\boldsymbol{x}} Y_n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \times \frac{\boldsymbol{x}}{|\boldsymbol{x}|},$$

$$\boldsymbol{A}_{2n}(\boldsymbol{x}) = \frac{|\boldsymbol{x}|}{\sqrt{\ell(\ell+1)}} \operatorname{grad}_{\boldsymbol{x}} Y_n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right),$$

$$\boldsymbol{A}_{3n}(\boldsymbol{x}) = Y_n\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \frac{\boldsymbol{x}}{|\boldsymbol{x}|}.$$

[8] See, e.g., Kristensson (2016).

Let $j_q(z), z \in \mathbb{C}$, be the Bessel function[9] of order $q \in \mathbb{Z}$.

For $n \in \mathbb{J}$, consider the functions

$$E_n^1(r) := j_\ell(kr),$$

$$F_n^2(r) := -\frac{j_\ell'\left(\frac{k}{\sqrt{\theta}}\right)\frac{k}{\sqrt{\theta}}}{j_\ell(k)\sqrt{\ell(\ell+1)}}\left(\frac{j_\ell(kr)}{r} + j_\ell'(kr)k\right) + \sqrt{\ell(\ell+1)}\,\frac{j_\ell\left(\frac{k}{\sqrt{\theta}}r\right)}{r},$$

and

$$F_n^3(r) := -\frac{j_\ell'\left(\frac{k}{\sqrt{\theta}}\right)\frac{k}{\sqrt{\theta}}}{j_\ell(k)}\,\frac{j_\ell(kr)}{r} + j_\ell'\left(\frac{k}{\sqrt{\theta}}r\right)\frac{k}{\sqrt{\theta}}\,.$$

By tedious calculations, see Ferraresso, Lamberti & $\mathbb{S}$ (2022), the following theorem can be proved.

[9] $j_q(z)$ the is the regular at $z=0$ solution of the Bessel ordinary differential equation $z^2 y_q''(z) + z\,y_q'(z) + \left(z^2 - q^2\right)y(z) = 0$, and is defined by the (convergent everywhere in $\mathbb{C}$) power series $\sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa}{\kappa!(q+\kappa)!}\left(\frac{z}{2}\right)^{q+2\kappa}$.

The Steklov eigenproblem in the case of the unit ball

Theorem

- *Let $k \neq 0$. Then, in the unit ball S^2 of $\mathbb{R}^3$, the eigenvalues and eigenfunctions of the Steklov problem (SM-θ) are given, for all $\ell \in \mathbb{N}$ and all $n \in \mathbb{J}$, by the two families*

$$\begin{cases} \lambda_n^{(1)} = -\dfrac{j_\ell'\left(\frac{k}{\sqrt{\theta}}\right) j_\ell(k) \frac{k^3}{\sqrt{\theta}}}{j_\ell\left(\frac{k}{\sqrt{\theta}}\right) j_\ell(k)\, \ell(\ell+1) - j_\ell'\left(\frac{k}{\sqrt{\theta}}\right) j_\ell(k) \frac{k}{\sqrt{\theta}} - j_\ell'\left(\frac{k}{\sqrt{\theta}}\right) j_\ell'(k)\, \frac{k^2}{\sqrt{\theta}}}, \\ \boldsymbol{F}_n = F_n^2 \boldsymbol{A}_{2n} + F_n^3 \boldsymbol{A}_{3n}, \end{cases} \tag{1}$$

$$\begin{cases} \lambda_n^{(2)} = -\left(1 + \dfrac{j_\ell'(k)}{j_\ell(k)} k\right), \\ \boldsymbol{E}_n = E_n^1 \boldsymbol{A}_{1n}. \end{cases} \tag{2}$$

- *Both $\lambda_n^{(1)}$ and $\lambda_n^{(2)}$ diverge to $-\infty$.*
- *It holds that* $\mathrm{div}\boldsymbol{F}_n \neq 0$ *and* $\mathrm{div}\boldsymbol{E}_n = 0$.

We conclude with the following

Remark

The denominators in (1) *and* (2) *vanish for certain values of* $k \neq 0$.

The corresponding k^2*'s are proved to coincide with the eigenvalues* A_n *of the Dirichlet problem* (*Dir*) *(slide 16), namely*

$$\begin{cases} \operatorname{curl}\operatorname{curl} \boldsymbol{E} - \theta \operatorname{grad}\operatorname{div} \boldsymbol{E} = A\,\boldsymbol{E}, & \text{in } \Omega, \\ \boldsymbol{E} = \mathbf{0}, & \text{on } \Gamma. \end{cases}$$

Hence, by our standing assumption

$$A_n < k^2 < A_{n+1},$$

both denominators are always different from 0.

The case $k = 0$

If we let $k \to 0$ in (1), (2), we obtain

$$\lim_{k\to 0} \lambda_n^{(1)} = -\frac{\ell\,(2\ell+3)\,\theta}{\ell\,(\theta+1)+1}\,,$$

$$\lim_{k\to 0} \lambda_n^{(2)} = -(\ell+1),$$

which (when $\theta = 1$) agree with the values computed by Raulot & Savo (2014), in their study of the spectrum of the Dirichlet-to-Neumann operator acting on forms of a Euclidean domain.

The above case corresponds to the limit in which the electric permittivity ε of a given material is 0 at a specific frequency, so the time-harmonic Maxwell's equations become

$$\operatorname{curl}\mathbf{E} - \mathrm{i}\,\omega\mu_{\mathrm{ENZ}}\,\mathbf{H} = \mathbf{0}\,, \quad \operatorname{curl}\mathbf{H} = \mathbf{0},$$

μ_{ENZ} being the magnetic permeability of the zero-permittivity material.
Wave propagation in this material can thus happen only with infinitely large phase velocity.
Such metamaterials are called ENZ (Epsilon-Near-Zero), and have very interesting applications, see, e.g., the book by Li, Zhou, He & Li (2022).

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ας είναι όλα NP επιλύσιμα!