The Unified Transform Method and Water Waves

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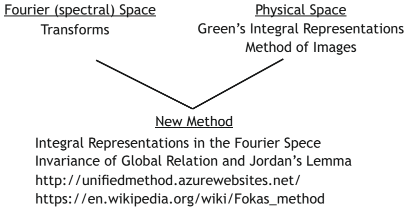
1747, d' Alembert and Euler: Separation of Variables
1807, Fourier: Transforms
1814, Cauchy: Analyticity
1828, Green: Green's Representations
1845, Kelvin: Images



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The heat equation

$$u_t = u_{xx}, \quad 0 < x < \infty, \;\; 0 < t < T, \quad T > 0,$$

Initial condition:

$$u(x,0) = u_0(x), \;\; 0 < x < \infty,$$

Boundary condition:

$$u(0,t) = g_0(t), \;\; 0 < t < T.$$

Classical Sine Transform method:

$$\text{PDE}[u(x,t)] \xrightarrow[\textit{Direct}]{ST} \text{ODE}[\hat{u}(\lambda,t)] \xrightarrow{\textit{Solve}} \hat{u}(\lambda,t) \xrightarrow[\textit{Inverse}]{ST} u(x,t)$$

The classical solution:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty e^{-\lambda^2 t}\sin(\lambda x)\Big[\int_0^\infty \sin(\lambda\xi)u_0(\xi)d\xi - \lambda\int_0^t e^{\lambda^2 s}g_0(s)ds\Big]d\lambda.$$

Disadvantages of the Traditional Transforms

1. Lack of uniform convergence at the boundaries (for inhomogeneous boundary conditions).
2. Not straightforward to verify that the solution representation actually solves the given BVP.
3. Not suitable for numerical computations.
4. Requires separability of PDE-domain-BCs. For example, cannot be applied to

$$\int_0^\infty K(x,t)u(x,t)dx = g(t).$$

5. Difficult to obtain appropriate transform.
6. Traditional transforms exist only for a very limited class of problems.

There are no x-transforms for the diffusion-convection equation

$$u_t = u_{xx} + u_x,$$

or the Stokes (written by Sir George Stokes)

$$u_t + u_x + u_{xxx} = 0.$$

Summary of Fokas Method

"Green + Fourier + Cauchy"

Three ingredients:

1. Global Relation - equation coupling transforms of boundary values.
2. Integral representation - solution given as integral of *transforms* of boundary values. ("Green + Fourier").
3. Symmetries which leave transforms of the boundary values invariant. Use information given by 1 to *either* find unknown boundary values *or* their contributions to 2.

In all but the simplest cases need to consider transforms of boundary values as functions in $\mathbb{C}$ ("Cauchy").

The heat equation via the Fokas Method

The heat equation can be written as a family of divergence forms:

$$\left(e^{-i\lambda x+\lambda^2 t}u\right)_t - \left(e^{-i\lambda x+\lambda^2 t}(u_x + i\lambda u)\right)_x = 0, \quad \lambda \in \mathbb{C}.$$

Green's Theorem: $$\int_{\partial\Omega} e^{-i\lambda x+\lambda^2 t}[udx + (u_x + i\lambda u)dt] = 0.$$

For the half line the Global Relation is

$$e^{\lambda^2 t}\hat{u}(\lambda, t) = \hat{u}_0(\lambda) - \tilde{g}_1(\lambda^2, t) - i\lambda\tilde{g}_0(\lambda^2, t), \quad \text{Im}\lambda \leq 0,$$

where

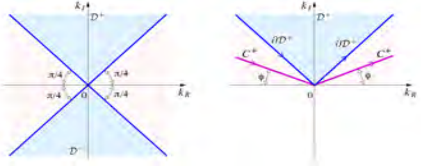
$$\hat{u}(\lambda, t) = \int_0^\infty e^{-i\lambda x}u(x, t)dx, \qquad \hat{u}_0(\lambda) = \int_0^\infty e^{-i\lambda x}u_0(x)dx, \ \text{Im}\lambda \leq 0,$$

$$\tilde{g}_j(\lambda, t) = \int_0^t e^{\lambda\tau}g_j(\tau)d\tau, \quad \text{with } g_1(t) = u_x(0, t), \ g_0(t) = u(0, t), \ t > 0.$$

First use of the GR:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(\lambda)d\lambda + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\left[\tilde{g}_1(\lambda^2,t) - i\lambda\tilde{g}_0(\lambda^2,t)\right]d\lambda.$$

The second integral can be deformed to the curve ∂D^+, defined by $\mathrm{Re}(\lambda^2) = 0,\ \mathrm{Im}\lambda > 0.$



Second use of the GR:

$$e^{\lambda^2 t}\hat{u}(-\lambda,t) = \hat{u}_0(-\lambda) - \tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t), \quad \mathrm{Im}\lambda \geq 0.$$

The solution takes the form

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[\hat{u}_0(-\lambda) + 2i\lambda\tilde{g}_0(\lambda^2,t)\right]d\lambda.$$

Also, we can replace $\tilde{g}_0$ with $G_0(\lambda^2) = \int_0^T e^{\lambda^2 s} g_0(s)ds, \quad \lambda \in \mathbb{C}.$

Consistent with the Ehrenpreis Principle:

$$u(x,t) = \int_L e^{i\lambda x-\lambda^2 t}d\rho(\lambda).$$

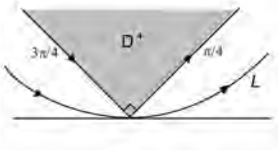
Numerical Implementation

Consider the heat equation with

$$u_0(x) = e^{-ax}, \quad g_0(t) = \cos(bt), \qquad a > 0, \; b > 0.$$

Then,

$$u(x,t) = \int_L e^{i\lambda x - \lambda^2 t} \left[\frac{1}{i\lambda + a} + \frac{1}{i\lambda - a} + i\lambda \left(\frac{1}{\lambda^2 + ib} + \frac{1}{\lambda^2 - ib} \right) \right] \frac{d\lambda}{2\pi}$$
$$- \int_L e^{i\lambda x} i\lambda \left(\frac{e^{ibt}}{\lambda^2 + ib} + \frac{e^{-ibt}}{\lambda^2 - ib} \right) \frac{d\lambda}{2\pi}.$$



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The last term can be evaluated explicitly, via Residue theory:

$$u(x,t) = \int_L e^{i\lambda x - \lambda^2 t} V(\lambda; a, b) \frac{d\lambda}{2\pi} + e^{-x\sqrt{\frac{b}{2}}} \cos\left(bt - x\sqrt{\frac{b}{2}}\right),$$

with $V(\lambda; a, b) = \frac{1}{i\lambda + a} + \frac{1}{i\lambda - a} + i\lambda\left(\frac{1}{\lambda^2 + ib} + \frac{1}{\lambda^2 - ib}\right)$.

Verification: Evaluating the above equation at $x = 0$ yields

$$u(0,t) = \int_L e^{-\lambda^2 t} V(\lambda; a, b) \frac{d\lambda}{2\pi} + \cos(bt) = \cos(bt).$$

By deforming the contour L to the real line and observing that $V(\lambda; a, b)$ is an odd function of λ, the above integral vanishes. Evaluating the previous equation at $t = 0$ yields

$$u(x,0) = \int_C e^{i\lambda x}\left(\frac{1}{i\lambda + a} + \frac{1}{i\lambda - a}\right)\frac{d\lambda}{2\pi}.$$

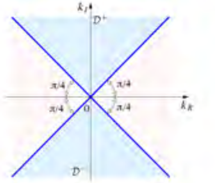
Then, Cauchy's theorem yields

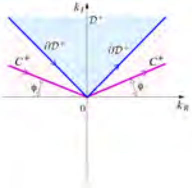
$$u(x,0) = \frac{2\pi i}{2\pi} e^{i(ia)x} \frac{1}{i} = e^{-ax}.$$

Solutions of linear second and third order PDES

$$u_t = u_{xx}, \qquad x > 0,\ t > 0$$

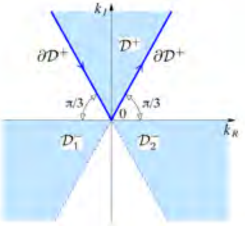
$$u(x,t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(\lambda)\frac{d\lambda}{2\pi} - \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left[\hat{u}_0(-\lambda) + 2i\lambda \tilde{g}_0(\lambda^2, t)\right]\frac{d\lambda}{2\pi}.$$





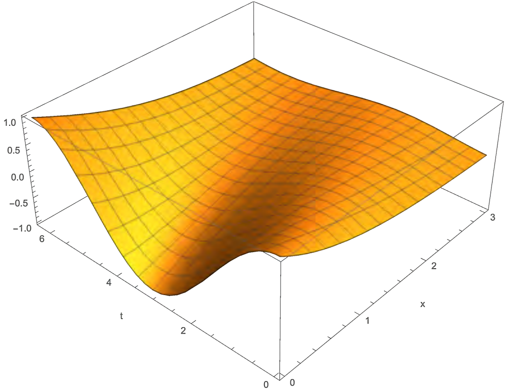
$$u_t + u_{xxx} = 0, \qquad x > 0,\ t > 0$$

$$u(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda x + i\lambda^3 t}\hat{u}_0(\lambda)\frac{d\lambda}{2\pi}$$
$$+ \int_{\partial D^+}\left\{e^{i\lambda x + i\lambda^3 t}\left[e^{\frac{2i\pi}{3}}\hat{u}_0\left(e^{\frac{2i\pi}{3}}\lambda\right) + e^{\frac{4i\pi}{3}}\hat{u}_0\left(e^{\frac{4i\pi}{3}}\lambda\right) - 3\lambda^2\tilde{g}_0(-i\lambda^3, t)\right]\right\}\frac{d\lambda}{2\pi}.$$



Heat Equation - Half Line
Dirichlet Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, +\infty),\ \ t \in (0, T), \\ u(x,0) = e^{-x}, & x \in (0, +\infty) \\ u(0,t) = \cos t, & t \in (0, T). \end{cases}$$



Fokas & Kaxiras, Modern Mathematical Methods for Scientists and Engineers, in press

Heat Equation - Half Line
Oblique Robin Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0, +\infty), \ \ t \in (0, T), \\ u(x, 0) = xe^{-4x}, & x \in (0, +\infty) \\ u_t(0, t) - 2u_x(0, t) + u(0, t) = \sin(5t), & t \in (0, T). \end{cases}$$

Solution:

$$u(x, t) = \int_L V(\lambda, x, t) \frac{d\lambda}{2\pi} + v(x, t),$$

where

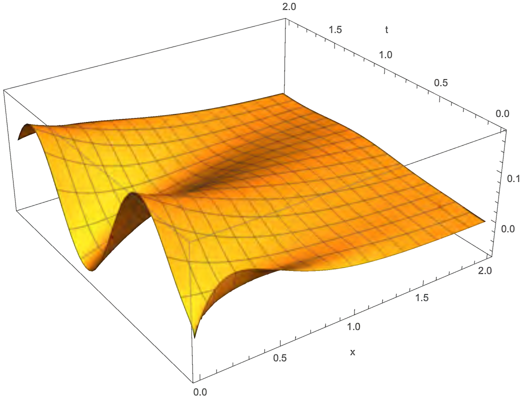
$$V(\lambda, x, t) = e^{i\lambda x - \lambda^2 t} \left(\frac{\frac{10i\lambda}{\lambda^4 + 25} + \frac{(\lambda - i)^2}{(\lambda + 4i)^2}}{(\lambda + i)^2} + \frac{1}{(4 + i\lambda)^2} \right),$$

and

$$v(x, t) = e^{-\sqrt{\frac{5}{2}}x} \frac{(\sqrt{10} + 1) \sin\left(5t - \sqrt{\frac{5}{2}}x\right) - (\sqrt{10} + 5) \cos\left(5t - \sqrt{\frac{5}{2}}x\right)}{\left(\sqrt{10} + 6\right)^2}.$$

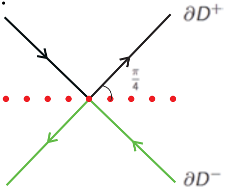
Heat Equation - Half Line
Oblique Robin Problem

Illustration of the solution:



Heat Equation - Finite Interval
General Dirichlet Problem

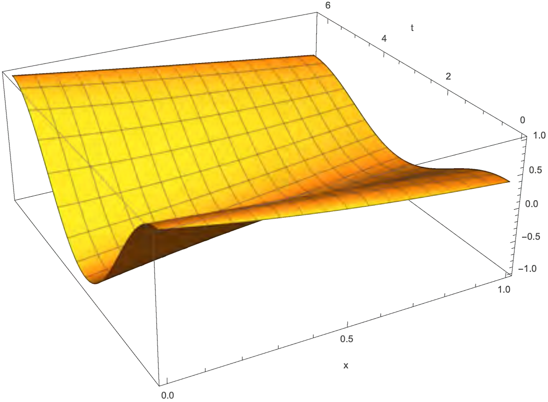
$$\begin{cases} u_t = u_{xx}, & x \in (0, L),\ \ t \in (0, T), \\ u(x, 0) = u_0(x), & x \in (0, L) \\ u(0, t) = g_0(t), \quad u(L, t) = h_0(t), & t \in (0, T). \end{cases}$$



$$u(x, t) = \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(\lambda) \frac{d\lambda}{2\pi}$$
$$- \int_{\partial D^+} \frac{2e^{-\lambda^2 t}}{e^{i\lambda L} - e^{-i\lambda L}} \Big\{ \sin(\lambda x) \Big[i e^{i\lambda L} \hat{u}_0(\lambda) - 2\lambda \tilde{h}_0\left(\lambda^2 t\right) \Big]$$
$$+ \sin[\lambda(L - x)] \Big[i \hat{u}_0(-\lambda) - 2\lambda \tilde{g}_0\left(\lambda^2 t\right) \Big] \Big\} \frac{d\lambda}{2\pi}$$

Heat Equation - Finite Interval
Specific example on Dirichlet Problem

$$\begin{cases} u_t = u_{xx}, & x \in (0,1),\ t \in (0,T), \\ u(x,0) = e^{-x}, & x \in (0,1) \\ u(0,t) = \cos t, \quad u(1,t) = \frac{1}{e}\cos t, & t \in (0,T). \end{cases}$$



The wave equation

$$u_{tt} - u_{xx} = 0, \qquad x > 0, \quad t > 0.$$

Dirichlet conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad x > 0,$$
$$u(0,t) = g_0(t), \qquad t > 0.$$

Solution in the Fourier plane:

$$u(x,t) = \int_{-\infty}^{\infty} e^{ikx} \left[\frac{\sin(kt)}{k} \hat{u}_1(k) + \cos(kt)\hat{u}_0(k) \right] \frac{dk}{2\pi}$$
$$- \int_{-\infty}^{\infty} e^{ikx} \left[\frac{\sin(kt)}{k} \hat{u}_1(-k) + \cos(kt)\hat{u}_0(-k) + 2ik\check{g}_0(k,t) \right] \frac{dk}{2\pi},$$

Solution in the physical plane:

$$u(x,t) = \frac{1}{2}u_0(x+t) + \frac{1}{2}\int_{|x-t|}^{x+t} u_1(\xi)d\xi + \begin{cases} \frac{1}{2}u_0(x-t), & x > t, \\ g_0(t-x) - \frac{1}{2}u_0(t-x), & x < t. \end{cases}$$

Laplace equation on a polygon

Let $u(x, y)$ satisfy the Laplace equation in the interior of a convex polygon characterized by $z_1, ..., z_n$. Then $u_z(z)$ satisfies

$$u_z(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} e^{i\lambda z}\hat{u}_j(\lambda)d\lambda, \qquad u_z = \frac{1}{2}(u_x - iu_y), \qquad z = x + iy,$$

where $\hat{u}_j(\lambda)$ is defined by

$$\hat{u}_j(\lambda) = \int_{z_j}^{z_{j+1}} e^{-i\lambda z}\left[\frac{\partial u}{\partial N} + \lambda u\frac{dz}{ds}\right]ds, \quad j = 1, ..., n, \qquad \lambda \in \mathbb{C}.$$

and the rays $\{l_j\}_1^n$ are defined by

$$l_j = \rho e^{i\theta_j}, \quad 0 < \rho < \infty, \quad \theta_j = -\arg(z_{j+1} - z_j), \quad j = 1, ..., n.$$

Furthemore, the following GR is valid:

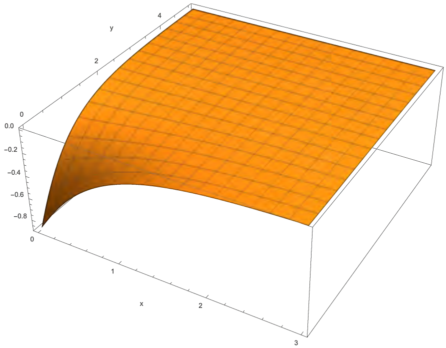
$$\sum_{j=1}^{n}\hat{u}_j(\lambda) = 0, \qquad \lambda \in \mathbb{C}.$$

Modified Helmholtz equation on the quarter plane

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) - u(x,y) = 0, & x > 0, \quad y > 0, \\ u_x(0,y) = e^{-2y}, \quad y > 0 \quad \text{and} \quad u_y(x,0) = e^{-3x}, & x > 0. \end{cases}$$

We obtain the following solution

$$u(x,y) = \int_0^{\infty e^{i\frac{\pi}{4}}} e^{\frac{i}{2}\left[\lambda(x+iy) - \frac{x-iy}{\lambda}\right]} \left(\frac{8\lambda}{\lambda^4 - 14\lambda^2 + 1} - \frac{12\lambda}{\lambda^4 + 34\lambda^2 + 1} \right) \frac{d\lambda}{\pi}.$$



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Water waves - The potential equation

Let the domain Ω_f be defined by

$$\Omega_f = \{-\infty < x < \infty, ; \quad -h < y < \eta(x,t); \quad t > 0\}.$$

Irrotational: vorticitity $\gamma = V_x - U_y = 0$.
Let ϕ denote the velocity potential, i.e. $\nabla\phi = (U, V)$.
The two unknown functions $\eta(x,t)$ and $\phi(x,y,t)$ satisfy the following equations:

$$\Delta\phi = 0 \quad \text{in} \quad \Omega_f,$$

$$\phi_y = 0 \quad \text{on} \quad y = -h,$$

$$\eta_t + \phi_x \eta_x = \phi_y \quad \text{on} \quad y = \eta,$$

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_y^2\right) + g\eta = 0 \quad \text{on} \quad y = \eta,$$

where g is the gravitational acceleration, and h is the constant unperturbed fluid depth.

The Global Relation

Introduce q which denotes the value of ϕ on the free surface, i.e.,

$$q(x,t) = \phi\left(x, \eta(x,t), t\right), \quad -\infty < x < \infty, \quad t > 0.$$

The global relation under appropriate transformations, yields a novel non-local equation coupling η and q,

$$\int_{-\infty}^{\infty} e^{ikx} \left\{ i\eta_t \cosh[k(\eta+h)] + q_x \sinh[k(\eta+h)] \right\} dx = 0, \;\; k \in \mathbb{R}, \;\; t > 0.$$

Furthermore, under the additional assumption of zero surface tension, equation the Bernoulli's law is rewritten

$$q_t + \frac{1}{2}\left(q_x\right)^2 + g\eta - \frac{\left[\eta_t + q_x \eta_x\right]^2}{2\left[1 + \left(\eta_x\right)^2\right]} = 0, \quad -\infty < x < \infty, \quad t > 0.$$

Periodic travelling waves

The above equation is a quadratic equation for q', thus we get

$$q' = -c + \sqrt{[1 + (\eta')^2](c^2 - 2g\eta)}.$$

Let $\Omega_p = \{-L < x < L, ; \quad -h < y < \eta(x,t); \quad t > 0\}$, and η and ϕ are $2L$-periodic functions in x.
We let $L = \pi$. Then we get

$$\int_{-\pi}^{\pi} e^{ikx} \Bigg[\left(1 - \sqrt{(1 + (\eta')^2) \left(1 - \frac{2g}{c^2}\eta \right)} \right) \sinh(k(\eta + h)) \\ + i\eta' \cosh[k(\eta + h)] \Bigg] dx = 0, \qquad \text{for all } k \in \mathbb{Z}$$

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Additional Applications

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From Nonlinear to Linear

$$u(x,t): \qquad iu_t + u_{xx} = 0.$$

Rewrite it as the compatibility condition $\left(M_x\right)_t - \left(M_t\right)_x = 0$, namely

$$\left[e^{i\lambda x + i\lambda^2 t} u\right]_t - i\left[e^{i\lambda x + i\lambda^2 t}(u_x + i\lambda u)\right]_x = 0, \quad \lambda \in \mathbb{C}.$$

By defining $M = e^{i\lambda x + i\lambda^2 t}\mu$, the associated Lax pair consists of the following two linear equations:

$$\begin{aligned} &\mu_x + i\lambda\mu = u, \\ &\mu_t + i\lambda^2\mu = iu_x + \lambda u, \qquad \lambda \in \mathbb{C}. \end{aligned}$$

Compare with the classical separation of variables:

$$u(x,t) = X(x;\lambda)T(t;\lambda) \quad \Longrightarrow \quad \begin{cases} X'' + \lambda^2 X = 0, \\ T' - i\lambda^2 T = 0. \end{cases}$$

A.S. Fokas and I.M. Gelfand, Integrability of linear and nonlinear evolution equations, and the associated nonlinear Fourier transforms (1994)

Asymptotics of the Neumann value for t-periodic data

Example

$$q(0,t) = ae^{i\omega t} + o(1), \quad a, \omega \in \mathbb{R}, \quad t \to \infty$$

Claim: For $\lambda = -1$ (focusing NLS)

$$q_x(0,t) = ce^{i\omega t} + o(1), \quad t \to \infty,$$
$$c = a\sqrt{\omega - a^2}, \qquad \omega \geq a^2,$$

or

$$c = ia\sqrt{|\omega| + 2a^2} \qquad \omega \leq -6a^2.$$

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