The Alber equation:
Landau damping & rogue waves in the ocean

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WAVES, PROBABILITIES AND MEMORIES
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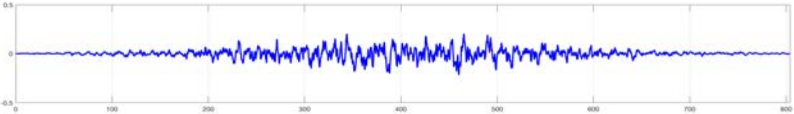
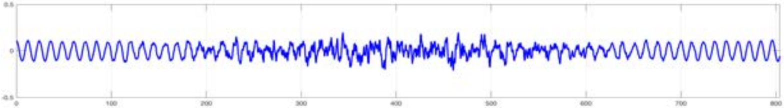
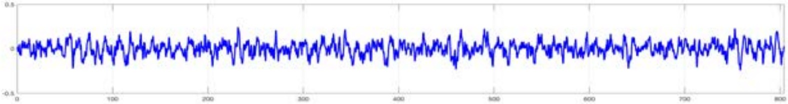


NLS + decay at infinity "always stable"



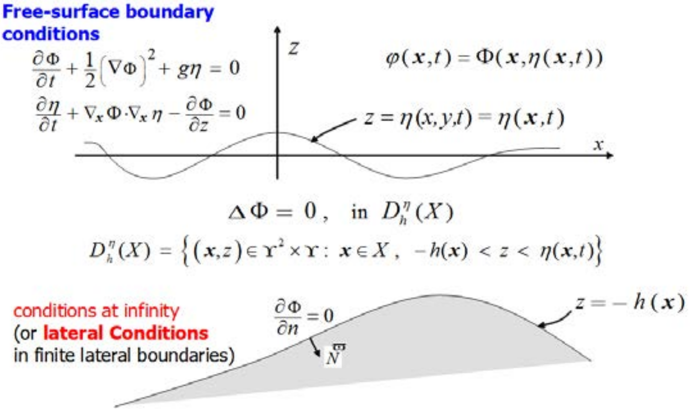
NLS + plane wave solution + localized perturbation
"modulation instability"



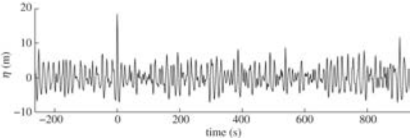
NLS + "homogeneous" background solution + localized perturbation
both stability & instability are possible

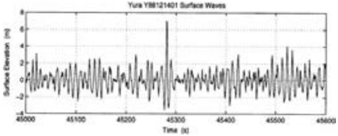
Classical ideal hydrodynamics – the Cauchy problem

- Assume **perfect knowledge of all initial fields**
- Assume bounded domain or **very simple behaviour at infinity**
- Recover **everything** at later times



What can the Cauchy problem say, e.g., about Rogue Waves?





Map from [E. Didenkulova, **Ocean & Coastal Management** 2020]

Find initial conditions that reproduce them (?)

Random inhomogeneous fields of nonlinear water waves

Background:

- I. E. Alber, **Proceedings of the Royal Society A** (1978)
  $\sim$ 240 citations in Google scholar
- D. R. Crawford, P. G. Saffman, H. C. Yuen, **Wave motion** (1980)
  $\sim$ 150 citations in Google scholar

This talk is based on:

- A. Athanassoulis, G. Athanassoulis and T. Sapsis,
  Journal of Ocean Engineering and Marine Energy (2017)
- A. Athanassoulis, **OMAE 2018**
- A. Athanassoulis, G. Athanassoulis, M. Ptashnyk and T. Sapsis,
  Kinetic and Related Models (2020)
- A. Athanassoulis and O. Gramstad, **Fluids** (2021)

Ocean waves modelling

- Gravity waves can be approximated by an **envelope** equation,

$$\eta(x,t) = \mathrm{Re}\left[u(x,t)e^{i(k_0\cdot x - \omega_0 \cdot t)}\right], \quad \omega_0 = \sqrt{gk_0},$$

  which turns out to be of **NLS** type

$$i\partial_t u + \underbrace{\frac{\sqrt{g}}{8k_0^{\frac{3}{2}}}}_{p/2} \Delta u + \underbrace{\frac{\sqrt{g}}{2}k_0^{\frac{5}{2}}}_{q/2} |u|^2 u = 0, \qquad k_0 \sim (0.01, 2).$$

  The order parameter is **slope**, $\beta \sim \frac{H_s k_0}{8\pi} < 0.1$.

- **Sea states** are typically **stationary & homogeneous** in mesoscales, characterised primarily through their **autocorrelation** / **spectrum**,

$$E\left[u(x,t)\overline{u}(x',t)\right] = \Gamma(x - x') + o(1),$$

$$\mathcal{F}_{y\to k}[\Gamma(y)] = P(k)$$

2^{nd} moments + Gaussian closure + Quasi-homogeneity

$$i\partial_t u + \frac{p}{2}\Delta u + \frac{q}{2}|u|^2 u = 0$$

$$R(\alpha,\beta,t) = E[u(\alpha,t)\bar{u}(\beta,t)]$$

$$\downarrow$$

$$i\partial_t R + \frac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right) R + \frac{q}{2} E\Big[u(\alpha,t)\bar{u}(\beta,t)\left[u(\alpha,t)\bar{u}(\alpha,t) - u(\beta,t)\bar{u}(\beta,t)\right]\Big] = 0$$

gaussian closure : Complex Isserlis Theorem,

$$E\Big[u(\alpha,t)\bar{u}(\alpha,t)u(\alpha,t)\bar{u}(\beta,t)\Big] = 2E\Big[u(\alpha,t)\bar{u}(\beta,t)\Big]E\Big[u(\alpha,t)\bar{u}(\alpha,t)\Big]$$

$$+$$

quasi-homogeneity : $R(\alpha,\beta,t) = \Gamma(\alpha-\beta) + \varepsilon\rho(\alpha,\beta,t)$

$$\downarrow$$

$$i\partial_t\rho + \frac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right)\rho + q\left[\Gamma(\alpha-\beta) + \varepsilon\rho(\alpha,\beta)\right]\left[\rho(\alpha,\alpha) - \rho(\beta,\beta)\right] = 0.$$

[I. E. Alber, PRSA 1978]

The Alber-Fourier equation (AF)

$\boldsymbol{f}$ is the inhomoegeneity, *in an appropriate frame of reference*. The spectrum $\boldsymbol{P}$ describes the homogeneous background.

$$\overbrace{\partial_t f - 4\pi^2 ipk\cdot Xf}^{\text{free-space}} + \underbrace{qi\left[P(k-\frac{X}{2}) - P(k+\frac{X}{2})\right]\breve{n}(X,t)}_{\text{interaction with spectrum}} +$$

$$+\underbrace{\varepsilon qi\int_s \breve{n}(s,t)\ \left[f(X-s,k-\frac{s}{2}) - f(X-s,k+\frac{s}{2})\right]ds}_{\text{self-interaction of inhomogeneity}} = 0.$$

$$\underbrace{\breve{n}(X,t) = \int_\xi f(X,\xi,t)d\xi,}_{\text{position density}} \qquad \underbrace{f(X,k,0) = f_0(X,k)}_{\text{initial data}}.$$

Linearised problem:

$$\partial_t f - 4\pi^2 ipk\cdot Xf + qi\Big[P\Big(k-\tfrac{X}{2}\Big) - P\Big(k+\tfrac{X}{2}\Big)\Big]\breve{n}(X,t) = 0,$$
$$f(X,k,0) = f_0(X,k) = \mathcal{F}^{-1}_{x\to X}[w_0], \qquad \breve{n}(X,t) = \int_{\mathbb{R}^d} f(X,\xi,t)d\xi.$$

Mild form:

$$f(X,k,t) - \overbrace{e^{4\pi^2 ipk\cdot Xt} f_0(X,k)}^{\text{free space solution}} = $$
$$= -qi\int_0^t e^{4\pi^2 ipk\cdot X(t-\tau)}\Big[P\Big(k-\tfrac{X}{2}\Big) - P\Big(k+\tfrac{X}{2}\Big)\Big]\breve{n}(X,\tau)d\tau,$$

Integrate in k:

$$\breve{n}(X,t) - \breve{n}_f(X,t) = \int_0^t h(X,t-\tau)\breve{n}(X,\tau)d\tau,$$

$$h(X,t) = 2q\,\sin(2\pi^2 pX^2 t)\breve{P}(2\pi pXt),$$

Laplace transform:

$$\widetilde{n}(X,\omega) = \widetilde{n}_f(X,\omega) + \widetilde{h}(X,\omega)\widetilde{n}(X,\omega) \quad \Rightarrow$$

$$\Rightarrow \quad \widetilde{n}(X,\omega) = \frac{1}{1-\widetilde{h}(X,\omega)}\widetilde{n}_f(X,\omega).$$

$$\tilde{n}(X,\omega) = \underbrace{\frac{1}{1-\tilde{h}(X,\omega)}}_{\text{transfer function}} \tilde{n}_f(X,\omega), \qquad \tilde{h} = \tilde{h}[P](X,\omega).$$

Penrose-type stability condition

$\exists \kappa > 0$ such that

$$\inf_{X, \mathrm{Re}\,\omega > 0} |1 - \tilde{h}(X,\omega)| \geqslant \kappa.$$

Then in principle we can invert $\tilde{n}(X,\omega) \mapsto \check{n}(X,t)$ and even get decay in time (with lots of technical work).

Theorem [A., Athanassoulis, Ptashnyk & Sapsis, KRM (2020)]

Let $P \in \mathcal{S}(\mathbb{R})$ be a background power spectrum of compact support which is **stable** in the sense that ***P* satisfies the Penrose stability condition**, and

$$\partial_t f - 4\pi^2 ipk \cdot Xf + qi\Big[P\Big(k - \tfrac{X}{2}\Big) - P\Big(k + \tfrac{X}{2}\Big)\Big]\breve{n}(X,t) = 0,$$
$$\breve{n}(X,t) = \int_\xi f(X,\xi,t)d\xi, \qquad f(X,k,0) = f_0(X,k) \in \Sigma^r$$

for $r \in \mathbb{N}$ large enough. Then $\exists C > 0$ s.t.

$$\|X\breve{n}\|_{L^2_{X,t}} = \|\partial_x n\|_{L^2_{x,t}} \leqslant C\frac{\kappa+1}{\kappa^2}\|f_0\|_{\Sigma^r} \tag{1}$$

and there exists a wave operator $\mathbb{W}$ s.t.

$$\lim_{t\to+\infty} |||f(X,k,t) - E(t)\mathbb{W}f_0||| = 0. \tag{2}$$

- Contrary to usual LD for Vlasov, **no mean-zero required for $w_0(x,k)$**. Thus we have linear LD for energy-carrying wavetrains interacting with a homogeneous background.
- Explains why free-space dynamics works so well in so many cases.

Working with the (in)stability condition

How do you check

$$\exists \kappa > 0 \qquad \inf_{X, \operatorname{Re}\omega > 0} |1 - \widetilde{h}(X,\omega)| \geqslant \kappa?$$

More or less through

$$\exists X_* \ \exists \operatorname{Re}\omega_* \geqslant 0 \qquad \widetilde{h}(X_*,\omega_*) = 1$$

A **nonlinear system of two equations in three unknowns**,

$$\operatorname{Re}\left(\widetilde{h}(X_*, a_* + ib_*)\right) = 1 \quad \text{and} \quad \operatorname{Im}\left(\widetilde{h}(X_*, a_* + ib_*)\right) = 0,$$

where of course $\omega_* = a_* + ib_*$.

Denote by $\mathbb{H}[f](t)$ the Hilbert transform,

$$\mathbb{H}[f](t) := p.v.\frac{1}{\pi}\int \frac{f(x)}{t-x}dx.$$

Theorem [A., Athanassoulis, Ptashnyk & Sapsis, KRM (2020)]

The following are equivalent:

(1). $$\inf_{\substack{\mathrm{Re}\,\omega>0,\\ X\in\mathbb{R}}} |1-\widetilde{h}(X,\omega)| = 0$$

(2). $$\exists\ X_* \in \mathbb{R}, \quad \Omega_* \in \mathbb{C}\backslash\mathbb{R} \quad : \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \mathbb{H}[D_{X_*}P](\overline{\Omega}_*) = \tfrac{4\pi p}{q}$$

or

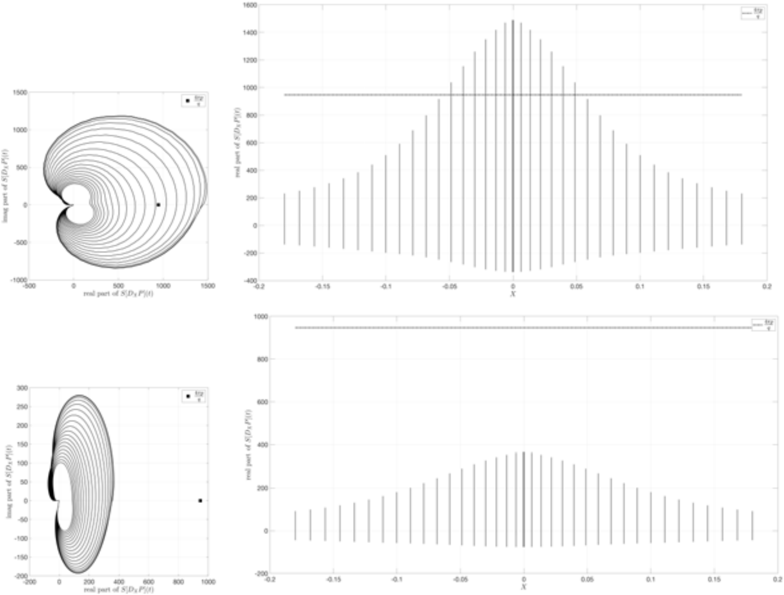
$$\exists\ X_*, \Omega_* \in \mathbb{R} \quad : \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \tfrac{4\pi p}{q} \ \text{ and } \ D_{X_*}P(\Omega_*) = 0.$$

(3). $d(\overline{\Gamma}, 4\pi p/q) = 0$, where $\mathbb{S}[f](t) = \mathbb{H}[f](t) - if(t)$ and

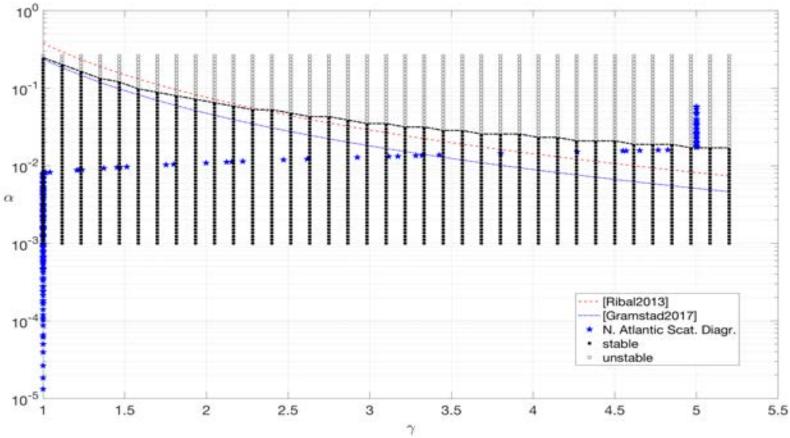
$$\Gamma_X := \{\mathbb{S}[D_X P(\cdot)](t),\ t\in\mathbb{R}\} \cup \{0\}, \qquad \mathring{\Gamma}_X = \{z\in\mathbb{C} | z \text{ enclosed by } \Gamma_X\},$$

$$\overline{\Gamma} := \overline{\bigcup_{X\in\mathbb{R}} \mathring{\Gamma}_X}.$$

The stability condition



Stability region for JONSWAP [AAPS KRM 2020]

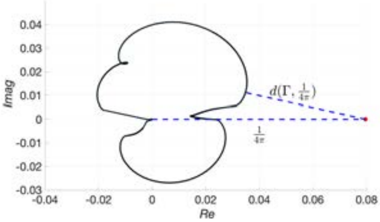


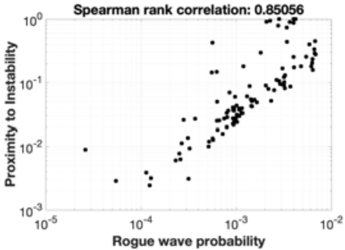
North Atlantic Scatter Diagram data from [DNV-GL, DNVGL-RP-C205: Environmental Conditions and Environmental Loads, Tech. Rep., August 2017]

Alber v. HOSM+Monte Carlo, [A., Gramstad, Fluids 2021]

Are there “more stable” and “less stable” spectra?

Nondimensional **“proximity to instability”** according to our analysis
versus
Monte Carlo simulations of the same sea state with a broadband solver.

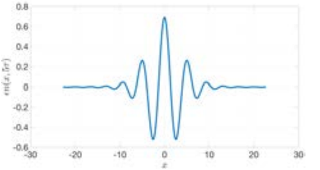




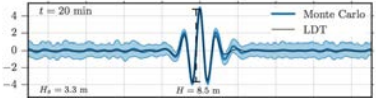
If there are unstable wavenumbers,

$$\widetilde{h}(X_*, \omega_*) = 1 \quad \text{for some } X_0 \in \mathbb{R},\ \operatorname{Re}\omega_* > 0$$

they give rise to **unstable modes**. These seem to successfully capture some inherent scalings for Rogue waves.



[A., Athanassoulis, Sapsis, JOEME (2017)]
Scalings of unstable modes



[Dematteis, Grafke, Vanden-Eijden, PNAS (2018)]
Monte Carlo + Large Deviations Theory
for fully Nonlinear simulations of mNLS

Summary

- Alber equation: 2^{nd} moment theory for NLS with stochastic initial data over a homogeneous background
- **Key features:**
  bifurcation between dispersion & modulation instability
  mathematical theory analogous to LD, many open questions still
- **Why ocean engineers care:**
  Finally takes into account metocean data!
  Results are plausible when compared to ocean data
  Provides insights on development of extreme events
- 2D, broadband versions of this are possible

Thank you very much!