In search of Fokker–Planck like description for dynamical systems driven by correlated noises

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Noise-induced transitions in dynamical systems

- Stochastically-driven dynamical systems appear in many fields, like energy harvesting, laser technology, turbulence, oncology etc.
- Environmental noises cannot be adequately described as delta-correlated white noise since they exhibit finite correlation time: colored i.e. correlated noise excitations.
- Computational cost: We can always perform Monte Carlo simulations, but they are computationally expensive, and also they are not analytic tools.

SDEs and the Fokker–Planck equation

Scalar stochastic differential equation (SDE) driven by potential V:

$$\dot{X}(t) = -V'(X(t)) + \sigma(X(t))\xi(t), \quad X(t_0) = x_0.$$

where $\xi(t)$ is the noise excitation and $\sigma(x)$ is the noise intensity.

For $\xi(t)$ Gaussian white noise, $C_\xi(t_1, t_2) = \delta(t_1 - t_2)$,
the classical Fokker–Planck equation is formulated

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\left[V'(x) - \frac{\varpi}{2}\sigma'(x)\sigma(x)\right]p(x,t)\right\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(x)p(x,t)\right],$$

for the response probability density function (PDF) $p(x,t)$.

- $\varpi = 0$: Itō interpretation
- $\varpi = 1$: Stratonovich interpretation

Fokker–Plank description for colored noise excitations

Can we formulate Fokker–Planck-like equations for SDEs under Gaussian correlated (colored) excitations, i.e. where $C_\xi(t_1, t_2)$ is a smooth function?

This was the topic of my PhD, where we derived PDF evolution equations of drift-diffusion form:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\left[V'(x) - \sigma'(x)\sigma(x)A_M(x,t;p)\right]p(x,t)\right\} + \\ + \frac{\partial^2}{\partial x^2}\left[\sigma^2(x)A_M(x,t;p)p(x,t)\right].$$

- These evolution equations are approximate (range of validity).
- Term A_M depends on the unknown PDF $p(x,t)$ via a particular response moment: nonlinear Fokker–Planck equation.
- For $A_M = 1/2$ we obtain the classical Fokker–Planck equation in the Stratonovich interpretation.

Nonlinear Fokker–Planck eq. for colored noise excitations

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\left[V'(x) - \sigma'(x)\sigma(x)A_M(x,t;p)\right]p(x,t)\right\} + \\ + \frac{\partial^2}{\partial x^2}\left[\sigma^2(x)A_M(x,t;p)p(x,t)\right].$$

with

$$A_M(x,t;p) = \sum_{m=0}^{M}\frac{D_m(t;p)}{m!}\left\{\zeta(x) - R(t)\right\}^m, \quad M = 0 \text{ or } 2$$

$$D_m(t;p) = \int_{t_0}^{t} C_\xi(t,s)\exp\left[\int_s^t R(u)\mathrm{d}u\right](t-s)^m\mathrm{d}s.$$

$$\zeta(x) = -\sigma(x)\left(\frac{V'(x)}{\sigma(x)}\right)', \quad R(t) = \mathbb{E}[\zeta(X)].$$

- $M = 0$: Hänggi's equation
- $M = 2$: our new equation (Mamis *et al.* 2019 *Proc. R. Soc. A*).

First example of colored noise: Ornstein-Uhlenbeck process

Let us specify now the noise excitation in SDE:

$$\dot{X}(t) = -V'(X(t)) + \sigma(X(t))\xi^{\mathrm{OU}}(t), \quad X(t_0) = x_0.$$

Colored noise $\xi^{\mathrm{OU}}(t)$ is the standard scalar Ornstein-Uhlenbeck Gaussian process, with zero mean and autocorrelation function

$$C_{\xi}^{\mathrm{OU}}(t_1, t_2) = \frac{1}{2s_{cor}} \exp\left(-\frac{|t_1 - t_2|}{s_{cor}}\right).$$

$s_{cor} > 0$ is the (finite) correlation time of OU noise.
Ornstein-Uhlenbeck noise is generated by the first-order, linear SDE:

$$\dot{\xi}^{\mathrm{OU}}(t) = -\frac{1}{s_{cor}}\xi^{\mathrm{OU}}(t) + \frac{1}{s_{cor}}\xi^{\mathrm{WN}}(t).$$

For $s_{cor} \to 0$ OU results in white noise.

Stationary distribution

Nonlinear Fokker-Planck:

$$\frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{\left[V'(x) - \sigma'(x)\sigma(x)A_M(x,t;p)\right]p(x,t)\right\} + \\ + \frac{\partial^2}{\partial x^2}\left[\sigma^2(x)A_M(x,t;p)p(x,t)\right].$$

The closed form of stationary response PDF $p_0(x)$:

$$p_0(x,R) = \frac{C(R)}{|\sigma(x)|A_M(x,R)}\exp\left(-\int^x \frac{V'(y)}{\sigma^2(y)A_M(y,R)}\mathrm{d}y\right),$$

where $R < 1/s_{cor}$ stationary value of $R(t)$, $\int^x \mathrm{d}y$ denotes the antiderivative, and $C(R)$ is the normalization factor.

Note that this closed form is not a solution by itself for the stationary nonlinear Fokker-Planck equation, since $p_0(x,R)$ depends also on the still undefined response moment R.

Iteration scheme for R

By using the definition of the response moment R,

$$R = \int_{\mathbb{R}} \zeta(x) p_0(x, R) \mathrm{d}x,$$

and then substituting the closed form for $p_0(x, R)$, we obtain the *self-consistency equation* of the form:

$$R = \mathcal{I}(R),$$

Thus, moment R is calculated via the iterative scheme; $R_{n+1} = \mathcal{I}(R_n)$, $n = 0, 1, \ldots$ As seen in the following numerical examples, this iteration scheme is rapidly convergent; for an error $\varepsilon_{tol} = 10^{-4}$, it converges after 4 iterations, on average.

Our semi-analytic form for the stationary response PDF

Thus, by using our **nonlinear Fokker–Planck equations**, we have an approximate, semi-analytic expression for the stationary response PDF of a **scalar SDE under colored noise excitation.**

$$\text{PDF form: } p_0(x, R) = \frac{C(R)}{|\sigma(x)| A_M(x, R)} \exp\left(-\int^x \frac{V'(y)}{\sigma^2(y) A_M(y, R)} \mathrm{d}y \right),$$

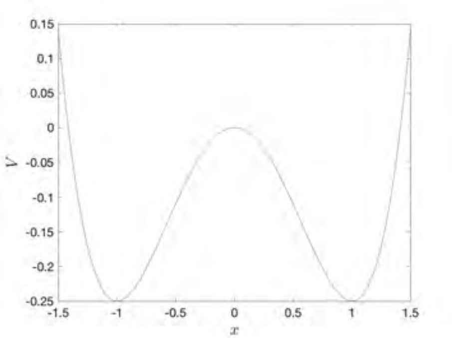
Rapidly-convergent iteration scheme for R: $R = \mathcal{I}(R)$.

This result can be used to determine the stationary response PDF without resorting to computationally expensive Monte Carlo simulations.

Bistable additively excited benchmark case

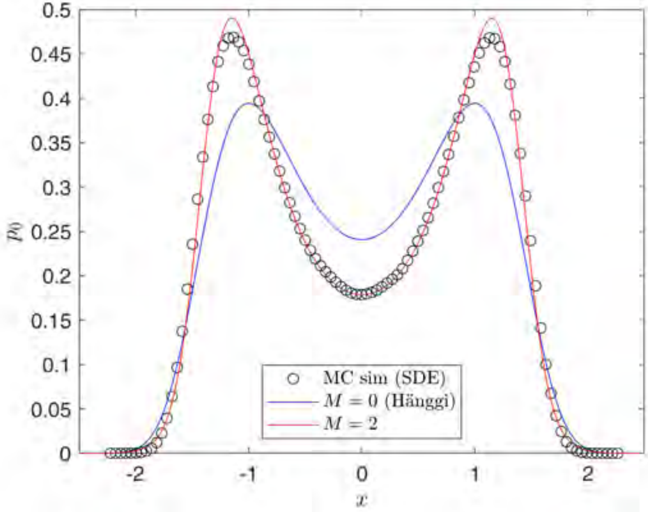
First example: SDE with symmetric bistable potential with wells at ± 1 under additive OU excitation:

$$\dot{X}(t) = -X^3(t) + X(t) + \sigma\xi(t).$$



Lyapunov time scale 0.5.

$\sigma = 1.2, \ s_{cor} = 0.25$

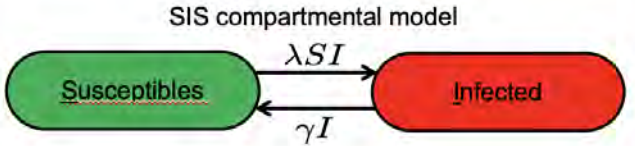


Compartmental Models of Epidemiology-SIS model

The SIS model is governed by the equations

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{\lambda}{N} SI + \gamma I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\lambda}{N} SI - \gamma I,$$

with the susceptible (S) and infected (I) compartments, N is the total population.



Compartmental Models of Epidemiology-SIS model

Constant population assumption and state variable $X = I/N \in [0,1]$:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \lambda X(1-X) - \gamma X$$

Basic reproduction number $R_0 = \lambda/\gamma$. Two equilibria: 0, $(\lambda-\gamma)/\lambda$.

- for $R_0 < 1$, equilibrium point 0 is stable (disease dies out)
- for $R_0 > 1$, equilibrium point 0 is unstable and $(\lambda-\gamma)/\lambda$ is stable. (endemic disease)

Lyapunov time scale $(\lambda-\gamma)^{-1}$.
We assume that the curing rate γ is a deterministic constant and the contact rate $\lambda(t) = \bar{\lambda} + \sigma\xi(t)$ is a stochastic process.

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \lambda X(1-X) - \gamma X + \sigma X(1-X)\xi(t)$$

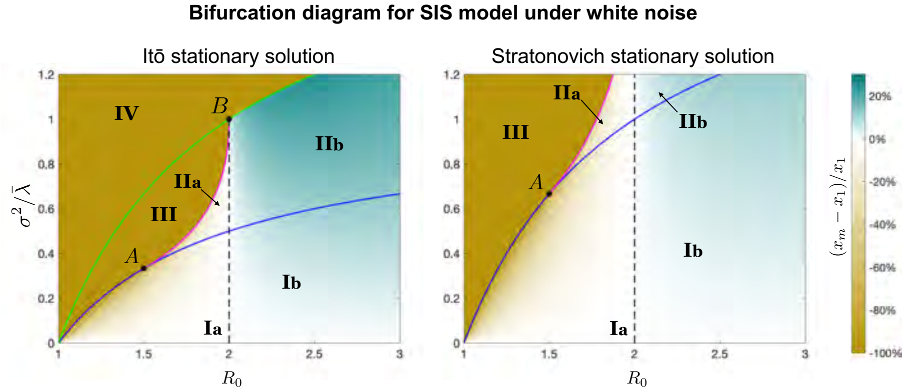
Stochastic SIS model, $\xi(t)$ is white noise

Stationary solution to the classical Fokker-Planck equation:

$$p_0(x) = \mathcal{N} x^{\frac{2\left(1-R_0^{-1}\right)}{(\sigma^2/\bar{\lambda})}-2+\varpi}(1-x)^{-\frac{2\left(1-R_0^{-1}\right)}{(\sigma^2/\bar{\lambda})}-2+\varpi}\exp\left(-\frac{2R_0^{-1}}{\left(\sigma^2/\bar{\lambda}\right)}\frac{1}{1-x}\right),$$

$\varpi = 0$: Itō solution, $\varpi = 1$: Stratonovich solution.

- Basic reproduction number R_0 of the underlying deterministic model.
- Relative noise level $\sigma^2/\bar{\lambda}$.

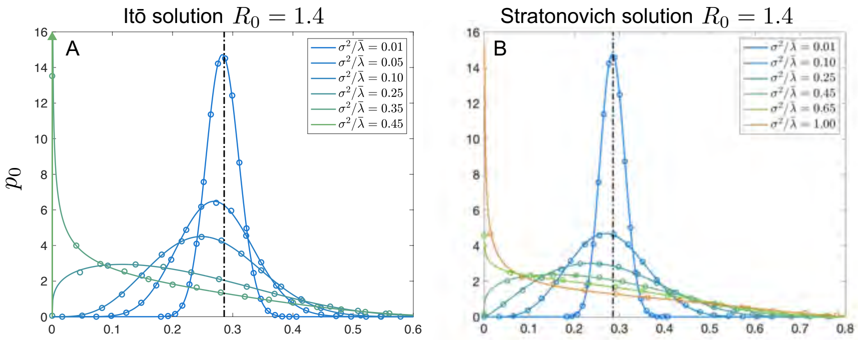


I: Unimodal with mode at a non-zero X. **II:** Bimodal with one mode at $X = 0$. **III:** Unimodal with mode at $X = 0$. **IV:** Delta function at $X = 0$.

Stochastic SIS model, $\xi(t)$ is white noise

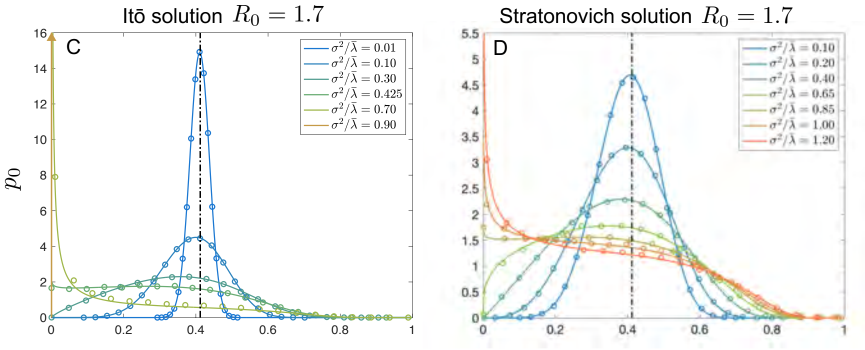
$$1 < R_0 < 1.5$$

Stationary response PDFs for SIS model under white noise



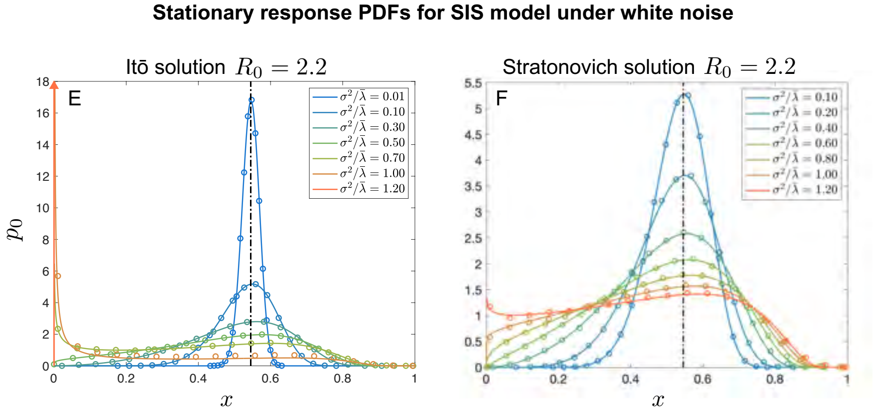
$$1.5 < R_0 < 2$$

Stationary response PDFs for SIS model under white noise



Stochastic SIS model, $\xi(t)$ is white noise

$$R_0 > 2$$



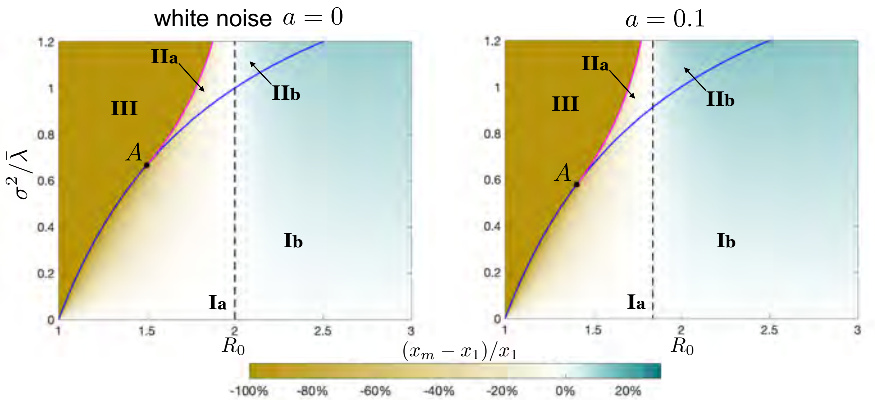
Stochastic SIS model, $\xi(t)$ is Ornstein-Uhlenbeck noise

In the case of stochastic SIS model, the stationary value of the response moment that makes our evolution equation a nonlinear Fokker-Planck is calculated analytically.

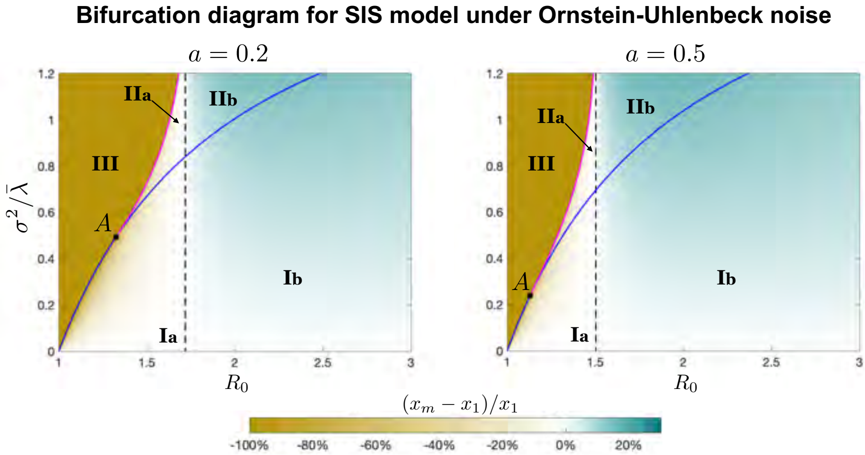
The stationary PDF is expressed in analytic form, and depends on three dimensionless parameters:

- Basic reproduction number R_0 of the underlying deterministic model.
- Relative noise level $\sigma^2/\bar{\lambda}$.
- Relative correlation time of OU noise $a = \tau(\bar{\lambda} - \gamma)$

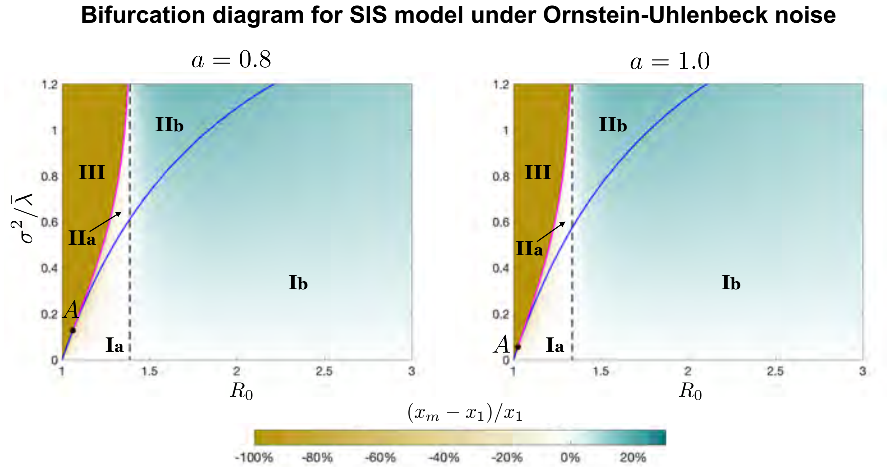
Bifurcation diagram for SIS model under Ornstein-Uhlenbeck noise



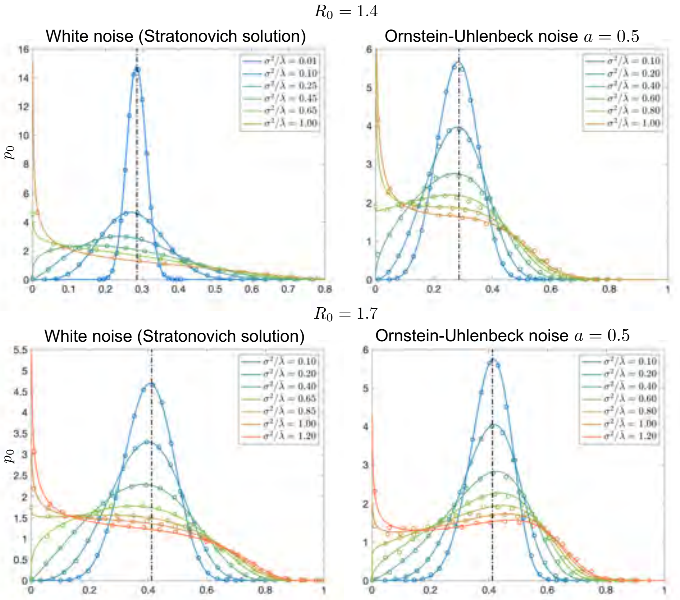
Stochastic SIS model, $\xi(t)$ is Ornstein-Uhlenbeck noise



Stochastic SIS model, $\xi(t)$ is Ornstein-Uhlenbeck noise

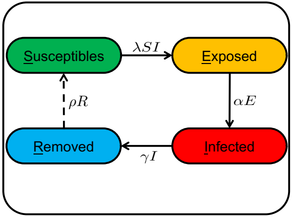


Correlated noise reduces the shift towards zero

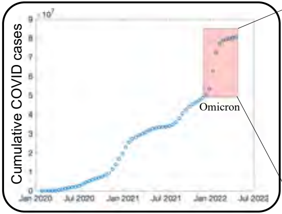


Stochastic SEIR model for COVID-19 pandemic

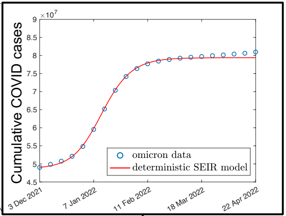
A. Compartmental epidemiological model



B. US data (Johns Hopkins database)

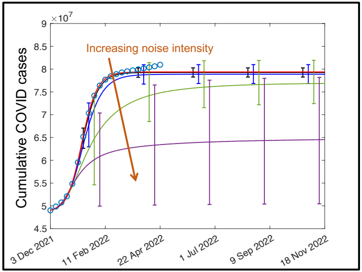


C. Fitting deterministic SEIR model to data

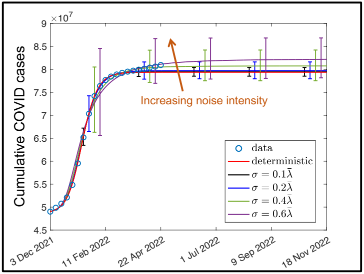


Stochastic perturbation in contact rate: $\lambda = \bar{\lambda} + \sigma\xi(t)$

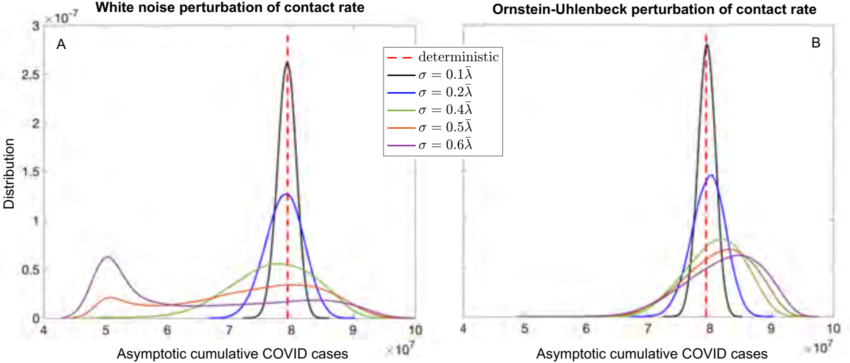
D. White noise: underestimates pandemic severity



E. Correlated noise: more accurate forecasts



Stochastic SEIR model for COVID-19 pandemic



Ornstein-Uhlenbeck correlation time: 1 week.

Summary

- New, approximate yet accurate nonlinear Fokker-Planck equation for SDEs under correlated noise.
- The effect of noise at dynamical systems is non-trivial, and exhibit richer behavior that undrlying deterministic problems.
- There are phenomena that arise only for correlated noise excitations.
- The response to correlated noise can be very different than the response to white noise: temporal correlations matter.

Thank you for your attention!
Questions?