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Variational modelling of rotational free-surface flows

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Literature review: the direction of Clebsch

- Starting point of the variational study of rotational flows is the **pioneering work of (Clebsch 1857; 1859)**. Especially in the latter work, he:
  - Showed that the velocity field may be expressed via the -now celebrated- Clebsch potentials (or variables) as

    $$\boldsymbol{u} = \nabla\varphi + m\nabla\psi$$

  - Recast the incompressible Euler equations in terms of the new variables φ, m, ψ
  - Provided an unconstrained variational principle for the new equations, with the pressure as the Lagrangian density
- Despite its significance for modern applications, his work was not recognized until decades later [(Grimberg and Tassi 2021)]

English translations were not available until August 2021!

Literature review: the direction of Clebsch

- It was 70 years later, when (Bateman 1929; 1944) extended Clebsch's approach to compressible flows
- Since then, several authors have been involved with the advantages and the limitations of Clebsch potentials

  [see e.g. (Eckart 1960), (Bretherton 1970), (Graham and Henyey 2000), (Wu, Ma, and Zhou 2006), (Kambe 2009), (Yoshida 2009), (Feldmeier 2020)]
- Also, Clebsch potentials have been used in numerous applications of various fields

  [see e.g. references in (Grimberg and Tassi 2021)]
- However, as far as nontrivial boundary conditions go, we were only able to find:
  - the suggestion of (Luke 1967), regarding the extension of the Clebsch-Bateman principle to free-surface flows, and
  - the very recent implementation of it by (Timokha 2015), for the problem of wave sloshing

Literature review: using Hamilton's principle

- In another line of work, the emphasis is on the transition:

$$\left.\begin{array}{c}\text{Variational principle in}\\ \text{Lagrangian description}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{Variational principle in}\\ \text{Eulerian description}\end{array}\right.$$

In the Lagrangian description, the variational formulation is a straightforward extension of Hamilton's principle

- In this direction, the primitive (energy) functional is:
  - rewritten in Eulerian variables
  - augmented with appropriate constraints
    (nature of the system, equivalence with Lagrangian counterpart)
- (Herivel 1955) made an initial attempt, imposing the constraints of the mass and entropy conservations:
  - Clebsch-like representation of the velocity field
  - Issue: necessarily irrotational flow for constant entropy

Literature review: using Hamilton's principle

- The issue was fixed by (Lin 1963), who noticed that, for an equivalent Eulerian principle, the variations must be carried out following the fluid motion [see, also, (Serrin 1959)]:

  Lin's conservation of identity (or constraint)

$$\frac{D\boldsymbol{a}(\boldsymbol{x},t)}{Dt} = 0, \qquad \boldsymbol{a}: \text{parcel labels}$$

additional constraint in the action functional

  - Extended or "classic" Clebsch representation, depending on the chosen number of conserved label components!

- Lin's constraint has been justified and/or used by many authors [e.g. (Seliger and Whitham 1968), (Bretherton 1970), (Van Saarloos 1981), (Bampi and Morro 1984), (Salmon 1988), (Fukagawa and Fujitani 2010)]

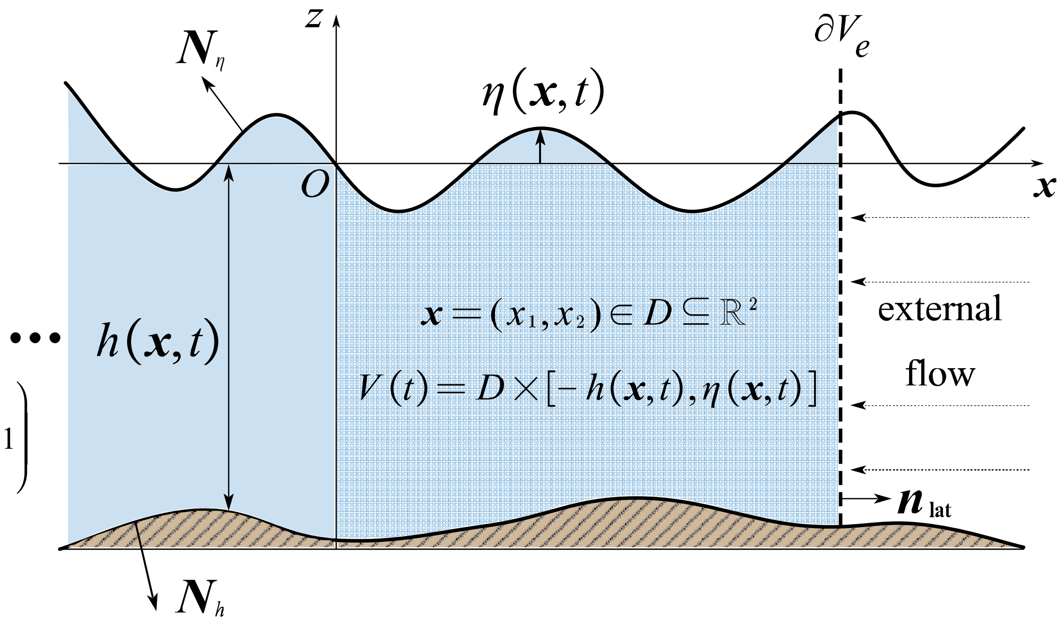
- Though, **boundary conditions seem to be overlooked in this direction, as well!**
  - The Eulerian free-surface flow is treated by (Berdichevsky 2009), but <u>kinematic conditions are a priori imposed</u>

Problem description & notation

- Ideal barotropic fluid with free surface, over moving seabed [density $\rho(\boldsymbol{x}, z, t)$, internal energy $E(\rho)$]
- Subject to applied pressure $\bar{p} = \bar{p}(\boldsymbol{x}, t)$ and conservative body (gravitational) forces given by potential $P = P(\boldsymbol{x}, z)$
- The (vertical) lateral boundary, ∂V, consists of two types:
  - ∂V_w: fixed rigid wall
  - ∂V_e: entrance (open) boundary

$$\boldsymbol{N}_\eta = \left(-\frac{\partial \eta}{\partial x_1}, -\frac{\partial \eta}{\partial x_2}, 1\right)$$

$$\boldsymbol{N}_h = \left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, -1\right)$$



Action functional of the problem

$$\tilde{\mathscr{S}}[\boldsymbol{a},\boldsymbol{u},\rho,\boldsymbol{A},k,\eta] = \underbrace{\int_{\mathrm{T}}\int_{D}\int_{-h}^{\eta} \rho\left(\frac{\boldsymbol{u}^2}{2} - E(\rho) - P\right) dz\, d\boldsymbol{x}\, dt}_{\text{primitive (energy) action functional}}$$

$$- \underbrace{\int_{\mathrm{T}}\int_{D}\int_{-h}^{\eta} \left\{ k\left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u})\right) + \rho\,\boldsymbol{A}\frac{D\boldsymbol{a}}{Dt}\right\} dz\, d\boldsymbol{x}\, dt}_{\text{constraints of mass \& identity conservations}}$$

$$+ \underbrace{\int_{\mathrm{T}}\int_{D} \overline{p}\,\eta\, d\boldsymbol{x}\, dt}_{\text{applied-pressure term}} + \underbrace{B.T.\Big|_{\partial V_e}^{(\mathrm{ext})}}_{\substack{\text{appropriate boundary terms on } \partial V_e \text{ for the} \\ \text{matching of the two flows (internal \& external)}}}$$

$B.T.\Big|_{\partial V_e}^{(\mathrm{ext})}$ is expected to be determined from the variational procedure!

Are the integral constraints enough ?

- Within the fluid domain, we may consider independent $\delta\rho$, $\delta\boldsymbol{a}$ & $\delta\boldsymbol{u}$, due to the mass and identity integral constraints

- The situation is different on the boundary:
  - No a priori reason to believe that such constraints, acting on the interior of the 3D fluid domain, work equally well on the lower-dimension boundary surface
  - In fact, if the variational procedure in the Eulerian formalism is attempted without additional constraints on $\delta\rho$, $\delta\boldsymbol{a}$ & $\delta\boldsymbol{u}$ on the boundary, implied by the Lagrangian nature of the boundary parcels, the derivation of any dynamic boundary condition is impossible (disintegration in separate parts)
  - On the free surface, the additional $\delta\eta$ occurs, whose relation with the rest of the variations should also be considered

Introducing a Lagrangian concept: virtual displacements

To overcome the issues on the boundary, we seek the relation between:

- the **Eulerian variations** ($\delta\rho$, $\delta\boldsymbol{a}$, $\delta\boldsymbol{u}$, $\delta\eta$), and
- the **virtual displacements** of the fluid parcels, $\delta_L \boldsymbol{X}$, which are the **natural variations of the system from the viewpoint of Analytical Dynamics**

For any Eulerian field of the flow, it may be shown that

$$\delta(\bullet) = \delta_L(\bullet) - (\delta_L \boldsymbol{X} \cdot \nabla)(\bullet) \quad (1)$$

where:

- $\delta_L(\bullet)$ is the Lagrangian variational operator, and
- $\delta_L \boldsymbol{X} = \delta_L \boldsymbol{X}\big(\boldsymbol{a}(\boldsymbol{x}, z, t), t\big)$ is the Eulerian representation of the virtual displacements

(Gelfand and Fomin 1963; Bretherton 1970; Mottaghi, et al. 2019)

Differential-variational constraints, in terms of $\delta_L \boldsymbol{X}$

Given that [(Bretherton 1970)]:

$$\delta_L \rho = -\rho(\nabla \cdot \delta_L \boldsymbol{X}), \quad \delta_L \boldsymbol{a} = 0, \quad \delta_L \boldsymbol{u} = \delta_L (D\boldsymbol{X}/Dt),$$

Eq. (1) yields the **differential-variational constraints**:

$$\begin{aligned} \delta\rho &= -\nabla\cdot(\rho\,\delta_L \boldsymbol{X}) \\ \delta\boldsymbol{a} &= -(\delta_L \boldsymbol{X}\cdot\nabla)\boldsymbol{a} \\ \delta\boldsymbol{u} &= \frac{D}{Dt}(\delta_L \boldsymbol{X}) - (\delta_L \boldsymbol{X}\cdot\nabla)\boldsymbol{u} \end{aligned} \qquad \text{(2a,b,c)}$$

- Point-wise conditions, applicable to any fluid parcel
- If used in the interior of the fluid domain, they render the integral constraints redundant and lead to the standard Euler equation

  ["hybrid" approach of (Bretherton 1970)]
- On the boundary, they should be combined with any additional constraints on $\delta_L \boldsymbol{X}$, implied by the boundary motion/dynamics

Virtual displacements on the boundaries

Free surface

- Arbitrary variations $\delta_L \boldsymbol{X}_\eta$ of the free-surface parcels
- If $S_\eta \equiv z - \eta(\boldsymbol{x}, t) = 0$ is the geometric representation of the free surface, then Eq. (1) leads to ($\delta_L S_\eta = 0$):

$$\boxed{\delta\eta = \delta_L \boldsymbol{X}_\eta \cdot \boldsymbol{N}_\eta}, \qquad \boldsymbol{N}_\eta = (-\partial_{x_1}\eta, -\partial_{x_2}\eta, 1)$$

Seabed

- Variations $\delta_L \boldsymbol{X}_h$ of the seabed parcels, for which:

$$\delta_L \boldsymbol{X}_h \cdot \boldsymbol{N}_h = 0, \qquad \boldsymbol{N}_h = (-\partial_{x_1} h, -\partial_{x_2} h, -1)$$

Lateral boundary

- Entrance boundary. Arbitrary variations $\delta_L \boldsymbol{X}_{\text{lat}}$
  (allowing for the matching of the dynamics of the two flows)
- Rigid wall. Variations $\delta_L \boldsymbol{X}_{\text{lat}}$, for which:

$$\delta_L \boldsymbol{X}_{\text{lat}} \cdot \boldsymbol{n}_{\text{lat}} = 0, \qquad \boldsymbol{n}_{\text{lat}} \text{ unit normal vector}$$

Back to $\tilde{\mathscr{S}}$: steps of the variational procedure

Based on the above remarks, the **variational equation** $\delta\tilde{\mathscr{S}} = 0$ (for the augmented action functional) is treated as follows:

Step 1: Calculation of the partial Gateaux derivatives

$$\delta_q \tilde{\mathscr{S}}[\boldsymbol{a}, \rho, \boldsymbol{A}, k, \boldsymbol{u}, \eta; \delta q], \qquad q \in \{\boldsymbol{a}, \rho, \boldsymbol{A}, k, \boldsymbol{u}, \eta\}$$

Step 2: Consideration of variations that vanish on the boundaries and derivation of the Euler-Lagrange equations corresponding to

$$\int_{\mathrm{T}} \int_{D} \int_{-h}^{\eta} (\cdots)\, \delta q \, dz \, d\boldsymbol{x} \, dt = 0, \qquad q \in \{\boldsymbol{a}, \rho, \boldsymbol{A}, k, \boldsymbol{u}\}$$

[δq independent inside V, due to the mass/identity constraints]

Step 3: Expression of the boundary remainder of $\delta\tilde{\mathscr{S}} = 0$ in terms of $\delta_L \boldsymbol{X}_\eta$, $\delta_L \boldsymbol{X}_h$ and $\delta_L \boldsymbol{X}_{\mathrm{lat}}$, via the differential-variational constraints

What was the motivation ?

Flashback to an early attempt ⟶ independent Eulerian variations on the boundary

- The kinematic conditions are correctly derived, but repetitively, for different "independent" variations
- The dynamic free-surface condition cannot be derived, unless one recognizes that at least $\delta\eta$ & $\delta u_{z=\eta}$ depend on each other via the respective differential-variational constraints
- The dynamic conditions on the entrance boundary cannot be derived without all the differential-variational constraints

This **redundancy** and **insufficiency** led to the introduction of the **differential-variational constraints** and to **Step 3**!

Variations in the interior of V: Euler-Lagrange equations

Conservations of mass & identity

$$\delta k: \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \qquad \delta \boldsymbol{A}: \quad \frac{D\boldsymbol{a}}{Dt} = 0$$

Conservation of Lagrange multipliers $\boldsymbol{A} = (A_1, A_2, A_3)$

$$\delta \boldsymbol{a}: \quad \frac{D\boldsymbol{A}}{Dt} = 0$$

Evolution of Lagrange multiplier k (pressure-related)

$$\delta \rho: \quad \frac{Dk}{Dt} = -\frac{\boldsymbol{u}^2}{2} + E(\rho) + \rho \frac{\partial E(\rho)}{\partial \rho} + P$$

Extended Clebsch representation for the velocity field

$$\delta \boldsymbol{u}: \quad \boldsymbol{u} = -\nabla k + \boldsymbol{A} \nabla \boldsymbol{a}$$

Remainder of the variational equation on the boundary

After treating the volume terms, the variational equation reduces to a boundary variational equation, of the form $\delta_b \tilde{\mathscr{S}} = 0$:

$$\int_{T}\int_{D}\left\{\left[\left(\frac{\partial\eta}{\partial t}-\boldsymbol{u}\,\boldsymbol{N}_{\eta}\right)(k\,\delta\rho+\rho\,\boldsymbol{A}\,\delta\boldsymbol{a})-\rho\,k\,\delta\boldsymbol{u}\,\boldsymbol{N}_{\eta}\right]_{z=\eta}\right.$$

$$\left.+\left[\overline{p}+\rho\left(\frac{\boldsymbol{u}^{2}}{2}-E-P\right)-k\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u})\right)-\rho\,\boldsymbol{A}\frac{D\boldsymbol{a}}{Dt}\right]_{z=\eta}\delta\eta\right\}d\boldsymbol{x}\,dt$$

Free surface

$$+\int_{T}\int_{D}\left\{\left(\frac{\partial h}{\partial t}-\boldsymbol{u}\,\boldsymbol{N}_{h}\right)(k\,\delta\rho+\rho\,\boldsymbol{A}\,\delta\boldsymbol{a})-\rho\,k\,\delta\boldsymbol{u}\,\boldsymbol{N}_{h}\right\}_{z=-h}d\boldsymbol{x}\,dt$$

Seabed

$$-\int_{T}\oint_{\partial D}\int_{-h}^{\eta}\left\{(k\,\delta\rho+\rho\boldsymbol{A}\,\delta\boldsymbol{a})\,\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}}+\rho\,k\,\delta\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}}\right\}dz\,dl\,dt+\delta B.T.\Big|_{\partial V_e}^{(\text{ext})}=0$$

Lateral boundary

Invoking the differential-variational constraints

- The variations $\delta\rho$, $\delta\boldsymbol{a}$ and $\delta\boldsymbol{u}$, in the boundary remainder, are substituted with the differential-variational constraints of Eqs. (2)
- Also, it can be easily verified that $\delta\eta = \delta_L \boldsymbol{X}_\eta \cdot \boldsymbol{N}_\eta$

Thus, the variational boundary remainder is rewritten in terms of $\delta_L \boldsymbol{X}_\eta$, $\delta_L \boldsymbol{X}_h$ and $\delta_L \boldsymbol{X}_{\text{lat}}$

Attention should be paid to the form of Eqs. (2) on the boundary, due to the nature of the Eulerian representations $\delta_L \boldsymbol{X}_b$:

E.g. $\delta_L \boldsymbol{X}_\eta$ is independent of z and, consequently,

$$[\delta\rho]_{z=\eta} = -[\nabla\rho]_{z=\eta} \cdot \delta_L \boldsymbol{X}_\eta - [\rho]_{z=\eta} \left(\nabla_2 \cdot (\delta_L X_{\eta,1}, \delta_L X_{\eta,2}) \right),$$

$$[\delta \boldsymbol{a}]_{z=\eta} = -[\nabla \boldsymbol{a}]_{z=\eta} \cdot \delta_L \boldsymbol{X}_\eta,$$

$$[\delta \boldsymbol{u}]_{z=\eta} \boldsymbol{N}_\eta = \frac{D_{2-\text{dim}}}{Dt} (\delta_L \boldsymbol{X}_\eta \boldsymbol{N}_\eta) - \left(\frac{\partial \boldsymbol{N}_\eta}{\partial t} + [\nabla(\boldsymbol{u} \boldsymbol{N}_\eta)]_{z=\eta} \right) \delta_L \boldsymbol{X}_\eta$$

Normal and tangential components of $\delta_L \boldsymbol{X}_b$

In the boundary variational equation, some terms are accompanied by $\delta_L \boldsymbol{X}_b$, and others by the normal components $\delta_L \boldsymbol{X}_b \, \boldsymbol{N}_b$.

Thus, to facilitate the analysis, we express $\delta_L \boldsymbol{X}_b$ as:

$$\delta_L \boldsymbol{X}_b = \delta_L \boldsymbol{X}_{b,\perp} + \delta_L \boldsymbol{X}_{b,\parallel}, \quad b \in \{\eta, h, \text{lat}\}$$

normal & tangential components

where:

- $\delta_L \boldsymbol{X}_{\{\eta,h\},\perp} = \delta B_{\{\eta,h\},\perp} \dfrac{\boldsymbol{N}_{\{\eta,h\}}}{\left\| \boldsymbol{N}_{\{\eta,h\}} \right\|^2}, \qquad \delta_L \boldsymbol{X}_{\text{lat},\perp} = \delta B_{\text{lat},\perp} \, \boldsymbol{n}_{\text{lat}}$
- $\delta_L \boldsymbol{X}_{b,\parallel} = \delta B_{b,1} \, \boldsymbol{T}_{b,1} + \delta B_{b,2} \, \boldsymbol{T}_{b,2}, \quad b \in \{\eta, h, \text{lat}\}$

  $\boldsymbol{T}_{b,\{1,2\}}$: tangent vectors - local basis of boundary's tangent plane

  (parametric representation of free surface/seabed & known ∂V_{lat})

Independent variations $\delta B_{b,\{\perp,1,2\}}$ in the place of $\delta_L X_{b,\{1,2,3\}}$

Free-surface term: tangential variations

Free surface

Considering, first, **tangential** variations:

$$\delta_L \boldsymbol{X}_{\eta,\parallel} = \text{arbitrary}, \qquad \delta_L \boldsymbol{X}_{\eta,\perp} = 0,$$

leads (after the required calculations) to the variational equation:

$$\int_T \int_D \left[\rho \left(\frac{\partial \eta}{\partial t} - \boldsymbol{u}\,\boldsymbol{N}_\eta \right) \left(-\nabla k + \boldsymbol{A}\,\nabla \boldsymbol{a} \right) \right]_{z=\eta} \boldsymbol{T}_{\eta,i}\, \delta B_{\eta,i}\, d\boldsymbol{x}\, dt = 0,$$

from which we obtain the free-surface **kinematic** condition:

$$\delta_L \boldsymbol{X}_{\eta,\parallel} : \quad \frac{\partial \eta}{\partial t} - \boldsymbol{u}\,\boldsymbol{N}_\eta = 0, \qquad z = \eta$$

Free-surface term: normal variations

Considering, next, **normal** variations:

$$\delta_L \boldsymbol{X}_{\eta,\perp} = \text{arbitrary}, \qquad \delta_L \boldsymbol{X}_{\eta,\parallel} = 0,$$

and using the derived kinematic condition, ultimately results in the variational equation:

$$\int_{\mathrm{T}}\int_{D} \left\{ \overline{p} + \rho \frac{Dk}{Dt} - \rho \boldsymbol{A} \frac{D\boldsymbol{a}}{Dt} + \rho \left(\frac{\boldsymbol{u}^2}{2} - E - P \right) \right\}_{z=\eta} \delta B_{\eta,\perp} \, d\boldsymbol{x} \, dt = 0$$

Accordingly, we obtain the free-surface **dynamic** condition:

$$\delta_L \boldsymbol{X}_{\eta,\perp} : \quad -\frac{Dk}{Dt} + \boldsymbol{A} \frac{D\boldsymbol{a}}{Dt} - \frac{\boldsymbol{u}^2}{2} + E + P = \frac{\overline{p}}{\rho}, \qquad z = \eta$$

Free-surface dynamic condition: Clebsch form

If we combine the free-surface boundary condition with the derived representation of the velocity, then the former becomes:

$$-\frac{\partial k}{\partial t} + \boldsymbol{A}\frac{\partial \boldsymbol{a}}{\partial t} + \frac{\boldsymbol{u}^2}{2} + E + P = \frac{\overline{p}}{\rho}, \qquad z = \eta$$

($\boldsymbol{u}$ is understood as a symbol for $-\nabla k + \boldsymbol{A}\nabla \boldsymbol{a}$)

- This expression is essentially the same as the Lagrangian density provided by (Clebsch 1859), with additional terms due to the inclusion of compressibility, conservative body forces and applied pressure
- In Sec. 9.3 of (Berdichevsky 2009), the same relation is derived for incompressible fluid, but the arbitrary addition of the zero term $\boldsymbol{A}(D\boldsymbol{a}/Dt)$ is required to the initial dynamic condition of his variational procedure

Seabed term & lateral boundary's rigid wall

Due to the nature of the seabed and the rigid wall, inducing the constraints $\delta_L \boldsymbol{X}_h \cdot \boldsymbol{N}_h = \delta_L \boldsymbol{X}_{\text{lat}} \cdot \boldsymbol{n}_{\text{lat}} = 0$, the virtual displacements on them are only tangential:

Moving seabed

$$\int_T \int_D \left[\rho \left(\frac{\partial h}{\partial t} - \boldsymbol{u}\,\boldsymbol{N}_h \right) \left(-\nabla k + \boldsymbol{A} \nabla \boldsymbol{a} \right) \right]_{z=-h} \boldsymbol{T}_{h,i}\, \delta B_{h,i}\, d\boldsymbol{x}\, dt = 0$$

$\Longrightarrow$ impermeability condition: $$\boxed{\frac{\partial h}{\partial t} - \boldsymbol{u}\,\boldsymbol{N}_h = 0, \qquad z = -h}$$

Rigid wall of the lateral boundary

$$\int_T \int_{\partial D_w} \int_{-h}^{\eta} \rho \left(\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}} \right) \left(-\nabla k + \boldsymbol{A} \nabla \boldsymbol{a} \right) \delta_L \boldsymbol{X}_{\text{lat},\parallel}\, dz\, dl\, dt = 0$$

$\Longrightarrow$ impermeability condition: $$\boxed{\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}} = 0, \qquad (\boldsymbol{x}, z) \in \partial V_w}$$

Open-boundary conditions: tangential variations

Open (entrance) boundary

Since it is an **open** boundary, we consider both tangential and normal variations, $\delta_L \boldsymbol{X}_{\text{lat},\parallel}$ and $\delta_L \boldsymbol{X}_{\text{lat},\perp}$, which yield **appropriate matching conditions** between the internal flow and the known external one

- For arbitrary **tangential** variations $\delta_L \boldsymbol{X}_{\text{lat},\parallel}$, and after the required calculations, we obtain the condition:

$$\rho(\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}})\underbrace{(-\nabla k + \boldsymbol{A}\nabla \boldsymbol{a})}_{\boldsymbol{u}} = B.T.\Big|_{\partial V_e,\parallel}^{(\text{ext})},$$

which constitutes the continuity of the momentum flux between the two parts of the flow

Open-boundary conditions: normal variations

- For **normal** variations $\delta_L \boldsymbol{X}_{\text{lat},\perp}$, using again standard algebraic manipulations, we derive the condition

$$\underbrace{\rho(\boldsymbol{u}\,\boldsymbol{n}_{\text{lat}})(-\nabla k + \boldsymbol{A}\nabla \boldsymbol{a})\,\boldsymbol{n}_{\text{lat}}}_{\text{normal component of momentum flux}} + \underbrace{\rho(Dk/Dt)}_{\text{pressure-related term}} = B.T.\Big|^{(\text{ext})}_{\partial V_e,\perp},$$

  which is interpreted as the continuity of the pressure between the two flows

- From terms on the line boundary of the entrance-boundary surface (boundary of co-dimension 2), we also obtain the matching of the velocity and gradient of the geometrical free surface and seabed

These open-boundary conditions appear for the first time, and, in retrospect, they seem very natural!

Where to, next?

- Despite the positive results, this is a very complicated and "inconvenient" variational principle, requiring several new concepts and lemmata in the process

- However, it also is a variational principle constructed on solid "physical ground", capable of producing the full equations of motion, along with the complete set of required boundary and matching conditions

- Thus, it equips us with all we need to implement our **next goal/step**, which is the construction of an:

> unconstrained action functional w.r.t. the velocity potentials (a priori velocity representation), with independent arguments whose variations lead to a similar complete set of equations

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