Coupled-Mode Theory for internal and free-surface waves

Conference in honour of Professor Gerassimos Athanassoulis

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École nationale supérieure de techniques avancées Bretagne (ENSTA Bretagne)

Collaborators

- Mathieu Mercier (Toulouse Institute of Fluid Mechanics, France)
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  Alexis Charalampompoulos (MIT, USA)

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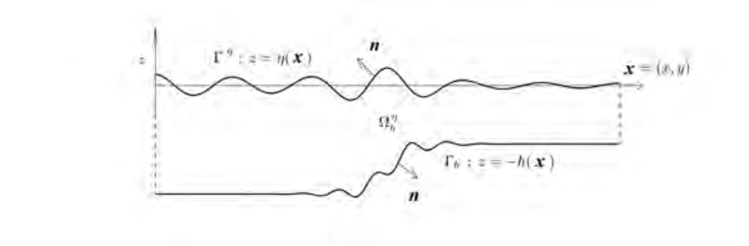
The Coupled-Mode approach

A semi-analytical method to solve waveguide problems



$$\text{Solve } \mathcal{P}: \begin{cases} \mathcal{D}U = F, & \text{in} \quad \Omega_h^{\eta} \\ \mathcal{B}_\eta U = f, & \text{on} \quad \Gamma^{\eta} \\ \mathcal{B}_h U = g, & \text{on} \quad \Gamma_h \end{cases},$$

$\mathcal{D}$ diff.operator

$\mathcal{B}_\eta, \mathcal{B}_h$ boundary operators

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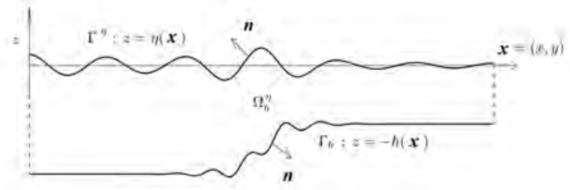

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Strategy

- Construct a representation $U^{\text{mod}}(\mathbf{x}, z, t) = \sum_n U_n(\mathbf{x}, t) Z_n(z; \mathbf{x}, t)$ in Ω_h^η, associated with a vertical Sturm-Liouville problem (reference waveguide) in $[-h(\mathbf{x}), \eta(\mathbf{x})]$

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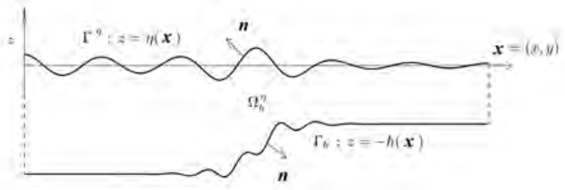

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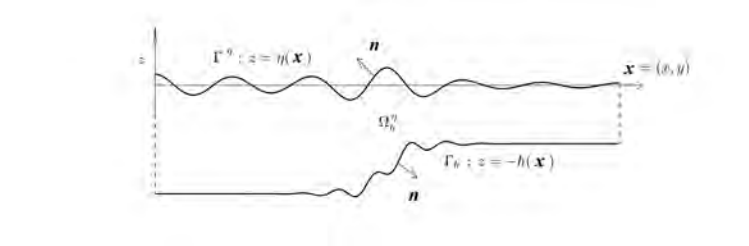

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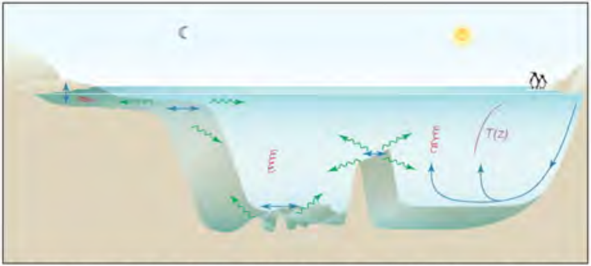
- Solve them numerically

Internal Tides

Internal tides in the ocean



Tides in the ocean. (Left) The to and fro of tidal currents generates internal waves at the edge of the continental shelf and over topographic features in the deep ocean. These internal waves can lead to turbulence and mixing. **(Right)** This mixing plays a role in maintaining a gradual transition between the sun-warmed surface layer of the ocean and the upwelling cold, dense water formed at high latitudes. *T*(*z*) denotes the temperature profile as a function of depth *z*.

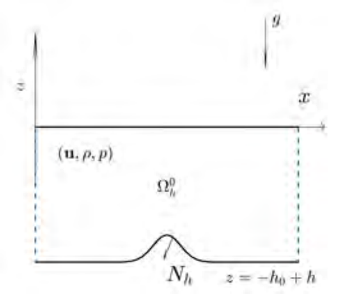
First observation Nansen (1893), Petterson (1908) and experiments (Ekman, 1904, Zeilon 1912, 1934)
Baines PG. 1973. The generation of internal tides by flat-bump topography. Deep-Sea Res.
Bell TH. 1975a. Lee waves in stratified flows with simple harmonic time dependence. J. Fluid Mech.
Bell TH. 1975b. Topographically generated internal waves in the open ocean. J. Geophys. Res.

linearized, inviscid Boussinesq equations

- The fluid is *ideal* and *stratified*

  & *linearization*

  & *f-plane approx.*



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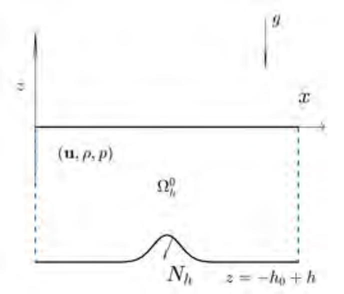

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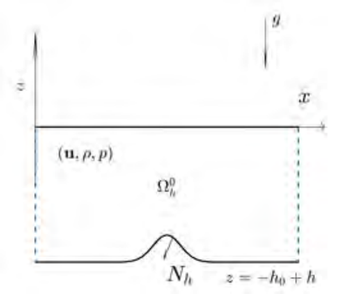

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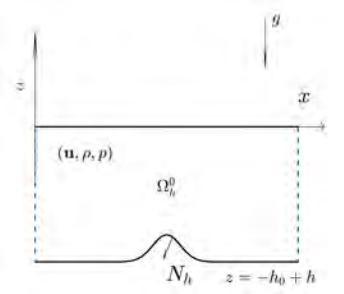

Physical Assumptions and approximations

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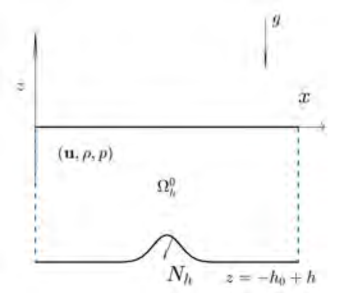

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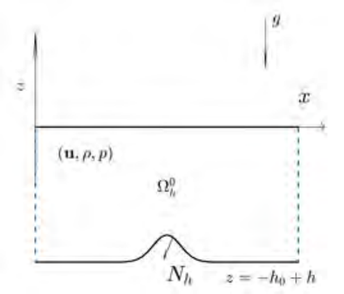
- Background barotropic tidal flow
- Background state $(p_0(z), \rho_0(z))$



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$$u_t = fv - \frac{1}{\overline{\rho}_0} p_x$$

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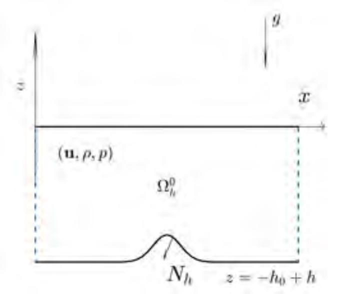

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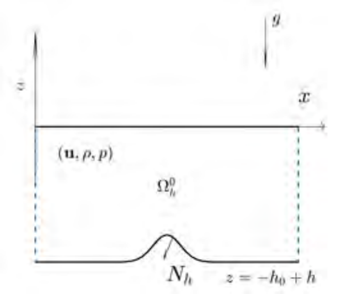

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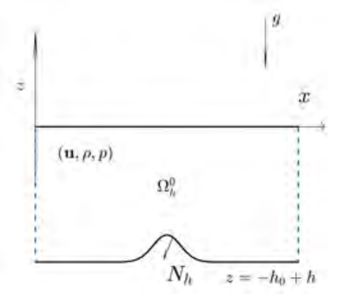

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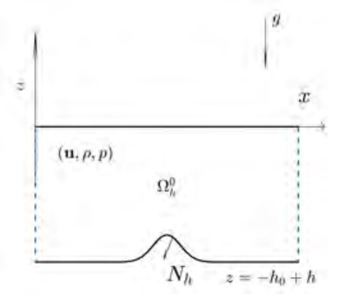

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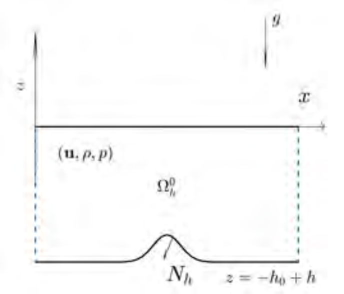

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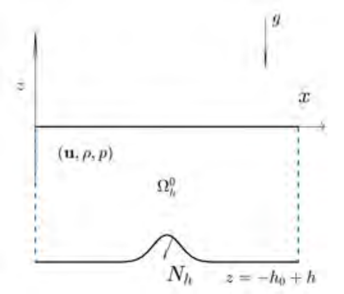

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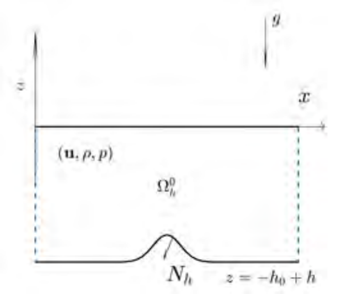

Further approximations allow analytical treatment (e.g. Fourier methods):

Weak Topography Approximation (WTA): $N_h \cdot (u, w) = 0$ on $z = -h_0$

Horizontally Uniform tidal flow: $(U, W) = (U_0 \cos \omega t, 0)$, $U_0 =$ cst

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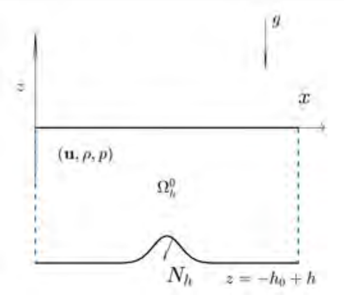

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Governing Equations • Stream function formulation

Introduce the buoyancy $b = -g\frac{\rho-\bar{\rho}_0}{\bar{\rho}_0}$ and the Brünt-Vaisällä frequency N, $N^2 = -g\frac{\rho_{0,z}}{\bar{\rho}_0}$

$$
\begin{aligned}
u_t - fv &= -p_x,\\
v_t + fu &= 0,\\
w_t &= -p_z + b,\\
b_t + N^2 w &= 0,\\
u_x + w_z &= 0,\\
N_h \cdot (u,w)|_{-h} &= 0,\\
w|_0 &= 0.\\
\int_{-h}^{0} u dz &= Q\cos\omega t
\end{aligned}
$$

$\psi(x,z,t) = \Re\{\phi(x,z)e^{-i\omega t}\}$ st $\psi|_{-h} = Q\cos\omega t$

$$
\mathcal{L}_\mu \phi \equiv \left(\frac{\partial^2}{\partial x^2} - \frac{1}{\mu^2}\frac{\partial^2}{\partial z^2}\right)\phi = 0,
$$

$$
\phi|_0 = 0, \quad \phi|_{-h} = Q.
$$

$\mu = \sqrt{\frac{N^2-\omega^2}{\omega^2-f^2}} > 0$, Q flow rate amplit.

$Q = 120$ m^2 s^{-1} corresponding e.g. to a barotropic velocity amplitude at $x \to -\infty$ of $U_0 = Q/h_0 = 4$ cm s^{-1} (resp. $U_0 = 4.46$ cm s^{-1}) and a depth $h_0 = 3$ km.
$\omega = 1.4 \times 10^{-4}$ s^{-1}, $f = 10^{-4}$ s^{-1} is the value around latitude 45°N, $N = 1.5 \times 10^{-3}$ s^{-1}, meaning that $\mu \approx 15.2$)

The internal tide generation problem

- The total flow is written $\phi = \underbrace{\Phi^{(0)}}_{\text{hydrostatic}} + \underbrace{\Phi^{\mathsf{NH}}}_{\text{non-hydrostatic}} + \phi^{\#} = \underbrace{\Phi^{(0)}}_{\text{hydrostatic}} + \underbrace{\Phi^{\mathsf{NH}} + \phi^{\#}}_{\phi^{\dagger}}$

BAINES PG. 1973. The generation of internal tides by flat-bump topography. *Deep-Sea Res.*

GARETT & GERKEMA 2006. On the Body-Force Term in Internal-Tide Generation. *J. Phys. Oceanogr.*

MAAS 2011, Topographies lacking tidal conversion, *J. Fluid Mech.*

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$$\left.\begin{aligned} \mathcal{L}_\mu \phi^\dagger &= -\mathcal{L}_\mu\left(-Q\frac{z}{h(x)}\right) \\ \phi^\dagger(x,0) &= 0 \\ \phi^\dagger(x,-h) &= 0 \end{aligned}\right\}\mathcal{P}$$

& radiations conditions
$\phi^\dagger \sim e^{\pm i k_n^\pm x}\sin(\kappa_n^\pm z)$, as $x \to \pm\infty$
$\kappa_n^\pm = \frac{n\pi}{-h_\pm}, \quad k_n^\pm = \frac{\kappa_n^\pm}{\mu}$

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- For general seamount and shelf topography, this problem has not been solved. Maas obtained special non-radiating solutions.
- Energy equation

$$\left[\int_{-h}^{0}\left\langle p^\dagger u^\dagger\right\rangle dz\right]_{-\infty}^{+\infty} = \left(1-\frac{\omega^2}{N^2}\right)\int_\Omega \left\langle W^{(0)} b^\dagger\right\rangle d\Omega.$$

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Normal mode decomposition

- The solution $\phi^\dagger$ is written in the form

$$\phi^\dagger(x,z) = \sum_{n=1}^{\infty} \phi_n(x) Z_n(z; h(x))$$

where the vertical functions Z_n are defined by the SL problem

$$\left.\begin{aligned} Z_{n,zz} + \tilde{\kappa}_n^2 Z_n &= 0, \quad -h(x) < z < 0, \\ Z_n &= 0, \quad z = 0 \\ Z_n &= 0, \quad z = -h(x), \end{aligned}\right\} \rightarrow (\tilde{\kappa}_n, Z_n) = \left(\frac{n\pi}{-h}, \sin(\tilde{\kappa}_n z)\right)$$

and the unknown modal amplitudes ϕ_n are defined by

$$\phi_n = \frac{2}{-h} \int_{-h}^{0} \phi^\dagger Z_n dz.$$

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- It can be shown that if $\phi^\dagger$ is sufficiently smooth, then $\phi_n = O(n^{-3})$, $\phi_{n,x} = O(n^{-3})$, and $\phi_{n,xx} = O(n^{-3})$.
- Through a Galerkin-type procedure, we derive equations on ϕ_n.

$$\phi_{m,xx} + \frac{\tilde{\kappa}_m^2}{\mu^2}\phi_m + \sum_{n=1}^{\infty} \frac{b_{mn}h_x}{h}\phi_{n,x} + \left(\frac{c_{mn}h_x^2}{h^2} + \frac{d_{mn}h_{xx}}{h}\right)\phi_n = Q\frac{2(-1)^m}{m\pi}h\left(\frac{1}{h}\right)_{xx}, \; m \geq 1,$$

$$\phi_{m,x} \pm ik_m^{\pm}\phi_m = 0, \quad \text{as} \quad x \to \pm\infty.$$

Energy conversion rate

$$C_{\pm} = \underline{\rho}_0 \frac{N^2 - \omega^2}{2\omega} \int_{-h_{\pm}}^{0} \Im\left\{\phi^{\dagger}\overline{\phi_x^{\dagger}}\right\} dz,$$

Baroclinic velocities

$$u^{\#} = -\Re\{\left(\phi^{\dagger} - \Phi^{\mathsf{NH}}\right)_z e^{-i\omega t}\}, \quad w^{\#} = \Re\{\left(\phi^{\dagger} - \Phi^{\mathsf{NH}}\right)_x e^{-i\omega t}\}$$

Weak topography approximation: $\delta = \frac{\max\{h\}}{h_0} \ll 1$, $\varepsilon = \mu \max\{|h_x|\} \ll 1$

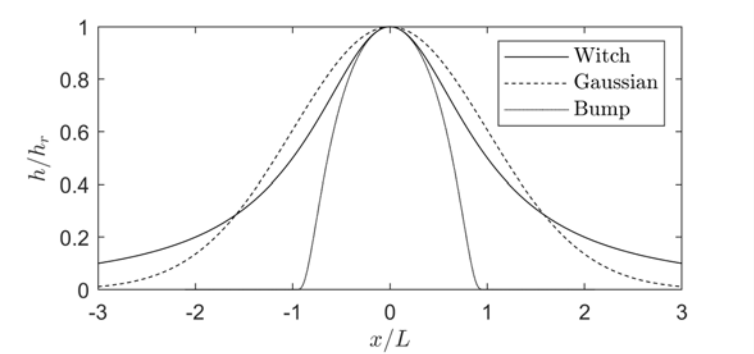
$$\mathcal{C}^{\mathsf{WTA}} = \frac{F_0}{h_0^2}\sum_{n=1}^{\infty} \lambda_n \hat{r}(\lambda_n)\bar{\hat{r}}(\lambda_n)\epsilon_{\lambda}, \quad F_0 = \frac{\underline{\rho}_0}{2\pi}\frac{\left[(N^2-\omega^2)(\omega^2-f^2)\right]^{1/2}}{\omega}U^2 h_0^2,$$

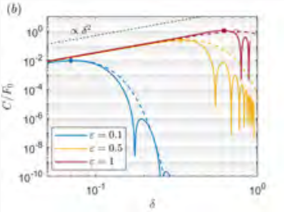
where $\lambda_n = n\pi/(\mu h_0)$, $\epsilon_{\lambda} = \lambda_n/n = \pi/(\mu h_0)$ and $\hat{r}(\xi) = \int_{-\infty}^{+\infty} \exp(-ix\xi)r(s)ds$.

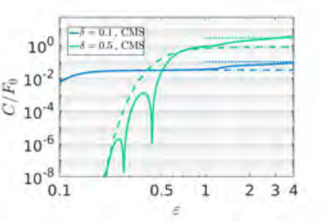
Llewellyn Smith & Young 2002, Conversion of the barotropic tide, *J. Phys. Oceanogr.*
St. Laurent et al. 2003, The generation of internal tides at abrupt topography, *Deep Sea Res.*
Ch. P et al. Internal tide generation from isolated seamounts and continental shelves (under revision)

$$h_{\mathsf{W}} = \frac{h_{\mathsf{r}}}{1 + \frac{x^2}{L^2}}, \quad h_{\mathsf{G}} = h_{\mathsf{r}} \exp\left(-\frac{x^2}{2L^2}\right), \quad h_{\mathsf{B}} = h_{\mathsf{r}} \exp\left(-\frac{1}{1 - \frac{x^2}{L^2}} + 1\right) \mathbb{1}_{(-L,L)},$$

Parameters: $\varepsilon = \mu \max\{|h_x|\}$ and $\delta = \dfrac{\max\{h\}}{h_0}$.

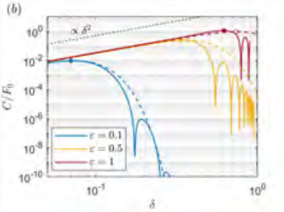


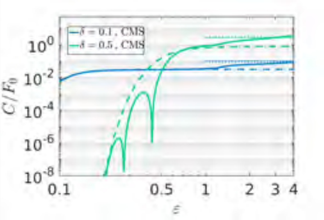




Llewellyn Smith & Young 2002, Conversion of the barotropic tide, *J. Phys. Oceanogr.*
St. Laurent et al. 2003, The generation of internal tides at abrupt topography, *Deep Sea Res.*
Maas 2011, Topographies lacking tidal conversion, *J. Fluid Mech.*

Results • Energy conversion rate





- Our calculations converge to WTA predictions as $\delta \to 0$

Llewellyn Smith & Young 2002, Conversion of the barotropic tide, *J. Phys. Oceanogr.*
St. Laurent et al. 2003, The generation of internal tides at abrupt topography, *Deep Sea Res.*
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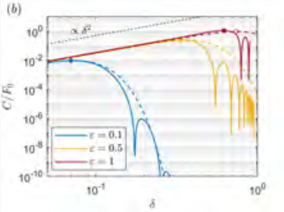

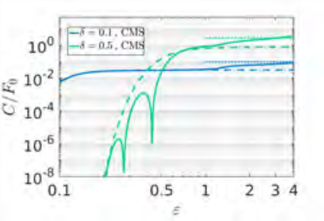

- Our calculations converge to WTA predictions as $\delta \to 0$
- (Gaussian) Error exceeds 20% for $\delta \approx 0.12$ ($\varepsilon = 0.1$), $\delta \approx 0.4$ ($\varepsilon = 0.5$), $\delta \approx 0.3$ ($\varepsilon = 1$)

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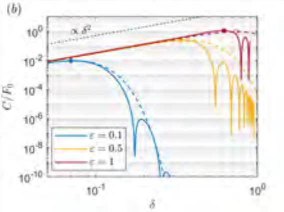
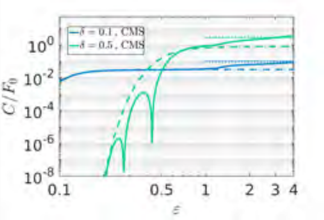
Results • Energy conversion rate





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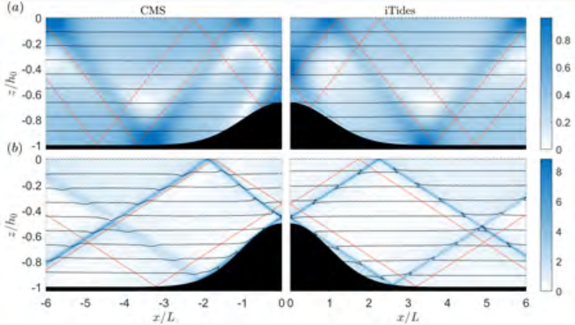
- (Gaussian) Error exceeds 20% for $\delta \approx 0.12$ $(\varepsilon = 0.1)$, $\delta \approx 0.4$ $(\varepsilon = 0.5)$, $\delta \approx 0.3$ $(\varepsilon = 1)$
- WTA does not predict non-radiating topographies in the subcritical regime

Llewellyn Smith & Young 2002, Conversion of the barotropic tide, *J. Phys. Oceanogr.*
St. Laurent et al. 2003, The generation of internal tides at abrupt topography, *Deep Sea Res.*
Maas 2011, Topographies lacking tidal conversion, *J. Fluid Mech.*

Video

Results



Echeverri, Peacock 2010, Internal tide generation by arbitrary two-dimensional topography, J. fluid. Mech

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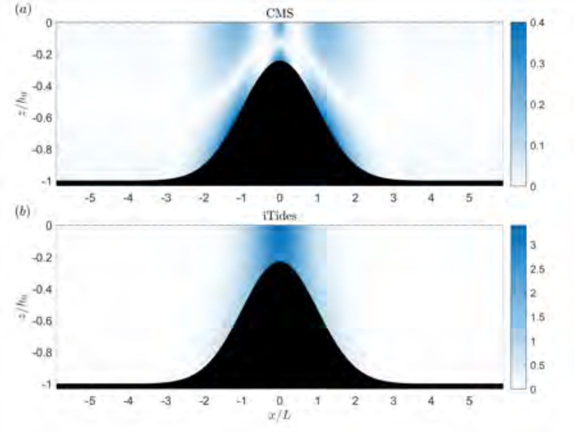

(a)
CMS
0
-0.2
-0.4
-0.6
-0.8
-1
z/h_0
-5 -4 -3 -2 -1 0 1 2 3 4 5
0.4
0.3
0.2
0.1
0
(b)
iTides
0
-0.2
-0.4
-0.6
-0.8
-1
z/h_0
-5 -4 -3 -2 -1 0 1 2 3 4 5
x/L
3
2.5
2
1.5
1
0.5
0

Water Waves

Physical Assumptions

- The fluid is *ideal* and *homogeneous*

Luke 1967, A variational principle for a fluid with a free surface, J. Fluid Mech.
Zakharov 1968, Stability of periodic waves on the surface..., J. Appl. Mech. Tech. Phys.
Craig & Sulem 1993, Numerical simulation of gravity waves, J. Comp. Phys.

Physical Assumptions

- The fluid is *ideal* and *homogeneous*
- The fluid is *incompressible*

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CRAIG & SULEM 1993, Numerical simulation of gravity waves, J. Comp. Phys.

Physical Assumptions

- The fluid is *ideal* and *homogeneous*
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- The flow is *irrotational*

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CRAIG & SULEM 1993, Numerical simulation of gravity waves, J. Comp. Phys.

Physical Assumptions

- The fluid is *ideal* and *homogeneous*
- The fluid is *incompressible*
- The flow is *irrotational*
- The free surface is impermeable

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Zakharov 1968, Stability of periodic waves on the surface..., J. Appl. Mech. Tech. Phys.
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Zakharov 1968, Stability of periodic waves on the surface..., J. Appl. Mech. Tech. Phys.
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Physical Assumptions

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- $\Delta\Phi = 0$, on $D_h^\eta(t)$
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Zakharov 1968, Stability of periodic waves on the surface..., J. Appl. Mech. Tech. Phys.
Craig & Sulem 1993, Numerical simulation of gravity waves, J. Comp. Phys.

Physical Assumptions

- $\partial_t \Phi + \frac{1}{2}(\nabla\Phi)^2 + gz = -\frac{1}{\rho}(P - P_{\text{atm}})$, on $D_h^\eta(t)$
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- $\Delta\Phi = 0$, on $D_h^\eta(t)$
- $\vec{V} = \nabla\Phi$
- $\partial_t \eta - N_\eta \cdot \nabla\Phi = 0$, on $\Gamma^\eta(t)$
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- $P = P_{\text{atm}}$, on $\Gamma_\eta(t)$

Luke's Variational Principle

$$\mathcal{S}[\eta, \Phi] = \int \int_X \int_{-h}^{\eta} \left[\partial_t \Phi + \frac{1}{2}(\nabla\Phi)^2 + gz \right] dz d\mathbf{x} dt.$$

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Hamiltonian formulation on $(\eta, \psi := \Phi(\mathbf{x}, \eta, t))$ and $\psi \to \mathcal{G}[\eta, h]\psi := N_\eta \cdot [\nabla\Phi]_{z=\eta}$

$$\partial_t \eta = \mathcal{G}[\eta, h]\psi$$

$$\partial_t \psi = -\frac{1}{2}|\nabla_{\mathbf{x}}\psi|^2 + \frac{\left(\mathcal{G}[\eta, h]\psi + \nabla_{\mathbf{x}}\psi \cdot \nabla_{\mathbf{x}}\eta\right)^2}{2\left(1 + |\nabla_{\mathbf{x}}\eta|^2\right)} - g\eta$$

(DtN) $\begin{cases} \Delta\Phi = 0 \\ N_h \cdot [\nabla\Phi]_{z=-h} = 0 \\ [\Phi]_{z=\eta} = \psi \end{cases}$

LUKE 1967, A variational principle for a fluid with a free surface, J. Fluid Mech.
ZAKHAROV 1968, Stability of periodic waves on the surface..., J. Appl. Mech. Tech. Phys.
CRAIG & SULEM 1993, Numerical simulation of gravity waves, J. Comp. Phys.

Consistent Coupled-Mode Theory

- $$\Phi(\mathbf{x},z,t) = \underbrace{\varphi_{-2}(\mathbf{x},t)Z_{-2}(z;\eta,h)}_{\text{Free Surface Mode}} + \underbrace{\varphi_{-1}(\mathbf{x},t)Z_{-1}(z;\eta,h)}_{\text{Sloping Bottom Mode}} + \sum_{n=0}^{\infty} \underbrace{\varphi_n(\mathbf{x},t)Z_n(z;\eta,h)}_{\text{Standard Modal Expansion}},$$

$$\varphi_{-1} = [\partial_z \Phi]_{-h}\,, \qquad Z_{-1}: \begin{cases} [\partial_z Z_{-1}]_\eta - \mu_0 [Z_{-1}]_\eta = 0 \\ [\partial_z Z_{-1}]_{-h} = 1 \end{cases}$$

$$\varphi_{-2} = [\partial_z \Phi - \mu_0 \Phi]_\eta\,, \qquad Z_{-2}: \begin{cases} [\partial_z Z_{-2}]_\eta - \mu_0 [Z_{-2}]_\eta = 1 \\ [\partial_z Z_{-2}]_{-h} = 0 \end{cases}$$

$$\varphi_n = \int_{-h}^{\eta} \underbrace{(\Phi - \varphi_{-1}Z_{-1} - \varphi_{-2}Z_{-2})}_{\Phi^*} Z_n dz, \qquad Z_n: \begin{cases} \partial_{zz} Z_n + k_n^2 Z_n = 0 \\ [\partial_z Z_n]_\eta - \mu_0 [Z_n]_\eta = 0 \\ [\partial_z Z_n]_{-h} = 0 \\ \mu_0 - k_0 \tanh(k_0(\eta+h)) = 0 \\ \mu_0 + k_n \tan(k_n(\eta+h)) = 0 \end{cases}$$

MASSEL 1993, Coast. Eng., PORTER & STAZIKER 1995, J. Fluid Mech.
ATHANASSOULIS & BELIBASSAKIS 1999, J. Fluid Mech., BEL. & ATH. 2011, Coast. Eng.
ATH. & PAPOUTSELLIS 2017, Exact semi-separation of variables in waveguides..., Proc. R. Soc. A

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- $O(n^{-4})$ decay and term-wise differentiability for any μ_0 (provided that $\Phi(x,\cdot)$ and its derivatives are $H^6(-h(x),\eta(x,t))$) and $\eta, h \in C^2$ ➥ Exact series expansion

Massel 1993, Coast. Eng., Porter & Staziker 1995, J. Fluid Mech.
Athanassoulis & Belibassakis 1999, J. Fluid Mech., Bel. & Ath. 2011, Coast. Eng.
Ath. & Papoutsellis 2017, Exact semi-separation of variables in waveguides..., Proc. R. Soc. A

Variational derivation of model equations

- extremize LUKE's functional $\mathcal{S}$ for functions of the form $\Phi = \sum_n \varphi_n Z_n(z;\eta,h) \equiv \boldsymbol{\phi Z}$
- Composition rule $\tilde{\mathcal{S}}[\eta,\boldsymbol{\phi}] = \mathcal{S}[\eta,\boldsymbol{\phi Z}] = \mathcal{S}\circ(\eta,\boldsymbol{\phi Z})$

$$\delta\varphi_m: \; 0 = \Big(\partial_t\eta - N_\eta\cdot[\nabla(\boldsymbol{\phi Z})]_\eta\Big)[Z_m]_\eta + \sum_n L_{mn}[\eta,h]\varphi_n, \text{ for all } m$$

$$\delta\eta: \; 0 = \big[\partial_t(\boldsymbol{\phi Z})\big]_\eta + g\eta + \frac{1}{2}\big[\nabla(\boldsymbol{\phi Z})\big]_\eta^2 - \sum_m\Big(\sum_n l_{mn}[\eta,h]\varphi_n\Big)\varphi_m$$
$$+\Big(-\partial_t\eta + N_\eta\cdot\big[\nabla(\boldsymbol{\phi Z})\big]_\eta\Big)\Big(\boldsymbol{\phi}\,[\partial_\eta\boldsymbol{Z}]_\eta\Big)$$

where

$$\sum_n L_{mn}(\eta,h)\varphi_n = \int_{-h}^{\eta}\Delta(\boldsymbol{\phi Z})Z_m dz - N_h\cdot[\nabla(\boldsymbol{\phi Z})]_{-h}[Z_m]_{-h}$$

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CH. P ET. AL 2019, Implementation of a fully nonlinear Hamiltonian Coupled-Mode Theory, and application to solitary wave problems over bathymetry. Eur. J. Mech. B Fluids

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- If $\boldsymbol{Z}$ are polynomials (from asymptotic expansions of Φ) ➠ high-order shallow approximations (Isobe-Kakinuma models and others)

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where

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$$\sum_n l_{mn}(\eta,h)\varphi_n = \int_{-h}^{\eta}\Delta(\boldsymbol{\phi Z})\partial_\eta Z_m dz - N_h\cdot[\nabla(\boldsymbol{\phi Z})]_{-h}[\partial_\eta Z_m]_{-h}$$

- If $\boldsymbol{Z}$ are polynomials (from asymptotic expansions of Φ) ➠ high-order shallow approximations (Isobe-Kakinuma models and others)
- Exact modal representation ➠ Exact modal reformulation on $(\eta,\boldsymbol{\phi})$

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where

$$\sum_n L_{mn}(\eta,h)\varphi_n = \int_{-h}^{\eta}\Delta(\boldsymbol{\phi Z})Z_m dz - N_h\cdot[\nabla(\boldsymbol{\phi Z})]_{-h}[Z_m]_{-h}$$

$$\sum_n l_{mn}(\eta,h)\varphi_n = \int_{-h}^{\eta}\Delta(\boldsymbol{\phi Z})\partial_\eta Z_m dz - N_h\cdot[\nabla(\boldsymbol{\phi Z})]_{-h}[\partial_\eta Z_m]_{-h}$$

- If $\boldsymbol{Z}$ are polynomials (from asymptotic expansions of Φ) ➠ high-order shallow approximations (Isobe-Kakinuma models and others)
- Exact modal representation ➠ Exact modal reformulation on $(\eta,\boldsymbol{\phi})$

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Variational derivation of the HCMS (Sketch)

$$\partial_t \eta - N_\eta \cdot \left[\nabla(\boldsymbol{\phi Z})\right]_\eta + \sum_{n=-2}^{\infty} L_{mn}[\eta, h]\varphi_n = 0, \text{ for all } m = -2, \ldots, \infty$$

$$\Downarrow$$

$$\begin{aligned} \partial_t \eta - N_\eta \cdot \left[\nabla(\boldsymbol{\phi Z})\right]_\eta + \sum_{n=-2}^{\infty} L_{m_* n}[\eta, h]\varphi_n &= 0, \\ \sum_{n=-2}^{\infty} L_{mn}[\eta, h]\varphi_n - \sum_{n=-2}^{\infty} L_{m_* n}[\eta, h]\varphi_n &= 0 \text{ for all } m = -2, \ldots, \infty \end{aligned}$$

$$\Downarrow$$

$$\partial_t \eta - N_\eta \cdot \left[\nabla(\boldsymbol{\phi Z})\right]_\eta = \Delta(\boldsymbol{\phi Z}) = N_h \cdot \left[\nabla(\boldsymbol{\phi Z})\right]_{-h} = 0$$

$$\Downarrow$$

$$\sum_{n=-2}^{\infty} L_{mn}[\eta, h]\varphi_n = 0, \quad \sum_{n=-2}^{\infty} l_{mn}[\eta, h]\varphi_n = 0 \text{ for all } m = -2, \ldots, \infty$$

The Hamiltonian Coupled Mode System (HCMS)

- Introducing $\psi = \sum_{n=-2}^{\infty} \varphi_n$, the water wave problem takes the form

$$\partial_t \eta = -\nabla_{\mathbf{x}} \eta \cdot \nabla_{\mathbf{x}} \psi + \left((\nabla_{\mathbf{x}} \eta)^2 + 1 \right) \left(h_0^{-1} \mathcal{F}_{-2}[\eta, h] \psi + \mu_0 \psi \right),$$
$$\partial_t \psi = -g\eta - \frac{1}{2} (\nabla_{\mathbf{x}} \psi)^2 + \frac{1}{2} \left((\nabla_{\mathbf{x}} \eta)^2 + 1 \right) \left(h_0^{-1} \mathcal{F}_{-2}[\eta, h] \psi + \mu_0 \psi \right)^2,$$

where $\mathcal{F}_{-2}[\eta, h]\psi := \varphi_{-2}$ is determined by solving a substrate

$$\begin{cases} \sum_{n=-2}^{\infty} (A_{mn} \Delta_{\mathbf{x}} + B_{mn} \cdot \nabla_{\mathbf{x}} + C_{mn}) \varphi_n = 0, \quad m \geq -2, \quad \mathbf{x} \in X, \\ \sum_{n=-2}^{\infty} \varphi_n = \psi, \quad \mathbf{x} \in X. \end{cases}$$

The Hamiltonian Coupled Mode System (HCMS)

- Introducing $\psi = \sum_{n=-2}^{\infty} \varphi_n$, the water wave problem takes the form

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$$\partial_t \psi = -g\eta - \frac{1}{2} (\nabla_{\mathbf{x}} \psi)^2 + \frac{1}{2} \left((\nabla_{\mathbf{x}} \eta)^2 + 1 \right) \left(h_0^{-1} \mathcal{F}_{-2}[\eta, h] \psi + \mu_0 \psi \right)^2,$$

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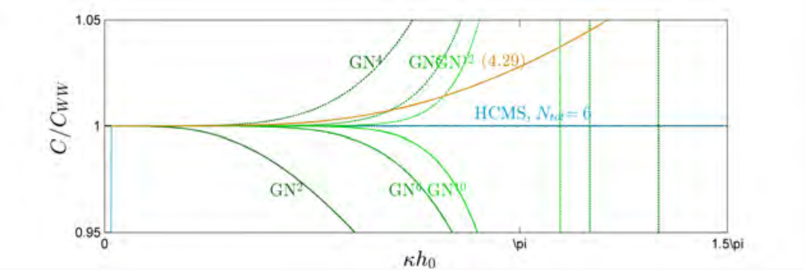
- DtN operator:

$$G[\eta, h]\psi = -\nabla_{\mathbf{x}} \eta \cdot \nabla_{\mathbf{x}} \psi + \left((\nabla_{\mathbf{x}} \eta)^2 + 1 \right) \left(h_0^{-1} \mathcal{F}_{-2}[\eta, h] \psi + \mu_0 \psi \right)$$

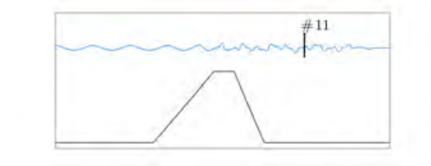
$$\frac{C_{\mathsf{WW}}}{\sqrt{gh_0}} = \left(\frac{\tanh(\kappa h_0)}{\kappa h_0}\right)^{1/2}$$

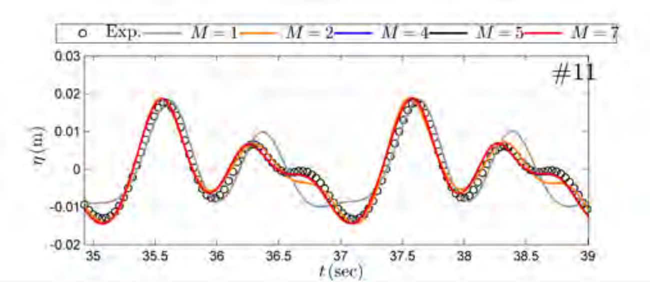
$$\frac{C_{\mathrm{WW}}}{\sqrt{gh_0}} = \left(\frac{\tanh(\kappa h_0)}{\kappa h_0}\right)^{1/2}$$



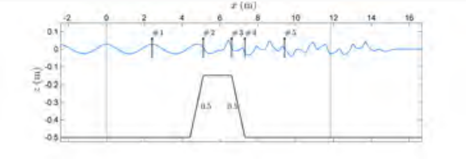
Validations

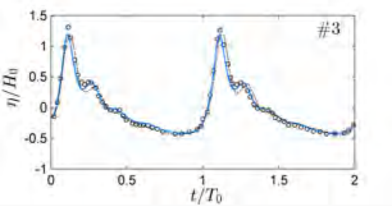
- L^2-Error for the DtN operator $\propto N^{-13/2}$, L^2-Error for Φ (flat bottom) $\propto N^{-7/2}$
- Steady Travelling Water Waves ($\lambda/h_0 = 0.5 - 28$ up to breaking)
- Harmonic generation (mild and strong bottom slope)
- Bragg reflection (sinusoidal bottom)
- Collision, reflection and shoaling of solitary waves
- Generation of solitary waves by abrupt bottom movement (tsunamis)
- 3D regular waves over an ellipsoidal bump and semicircular shoal
- Extreme waves over flat and variable depth (nonlinear wavegroups)
- Solitary waves over undulating bathymetry, abrupt deepenings and trenches


#11


Exp.
M = 1
M = 2
M = 4
M = 5
M = 7
#11
η (m)
t (sec)
0.03
0.02
0.01
0
-0.01
-0.02
35
35.5
36
36.5
37
37.5
38
38.5
39

Harmonic Generation





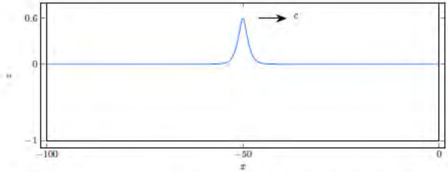
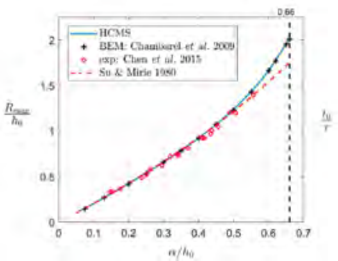
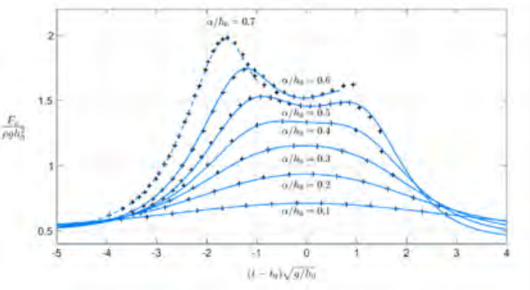


Figure 8.5: Initial free surface and configuration of the domain for the study of the reflection of solitary waves on a vertical wall.





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Semi-circular shoal

Video

Semi-circular shoal