COUPLED MODE SYSTEMS FOR HEAT CONDUCTION IN COMPOSITE MEDIA

Theodosios K. Papathanasiou

Department of Civil Engineering

Aston University
BIRMINGHAM UK

OUTLINE

- Background
- Heat Transfer in Pavement Repair Processes
- Coupled Mode Expansion Based FEM
- Vertical Eigenvalue Problem (Composite Media)
- Semi-explicit Formulas for the Eigenvalues (and a few slides about Water Waves)

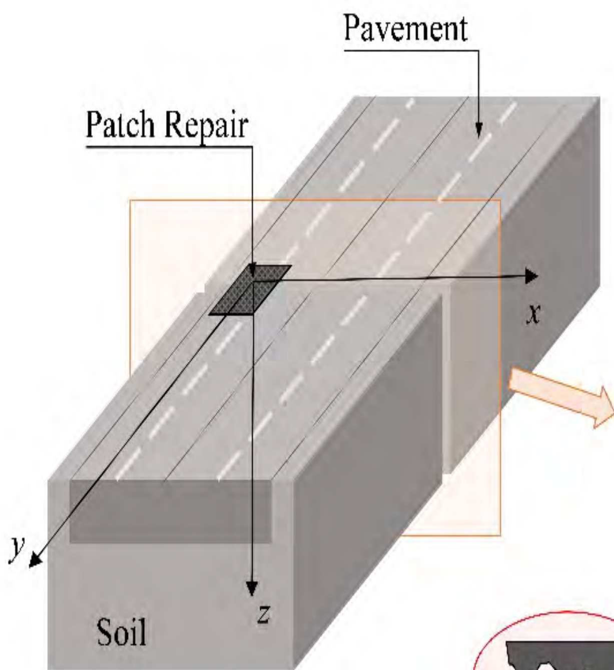
Heat Transfer in Pavement Repair Processes

- **Road deterioration due to potholing** has severe impact on smooth traffic flow and could be the cause of accidents, vehicle damage and road user distress. According to the **Annual Local Authority Road Maintenance Survey ALARM**, in 2021 alone, there are over 1.5m potholes being filled in with total cost that exceeds £93m (https://www.asphaltuk.org/wp-content/uploads/ALARM-survey-2021-FINAL.pdf).

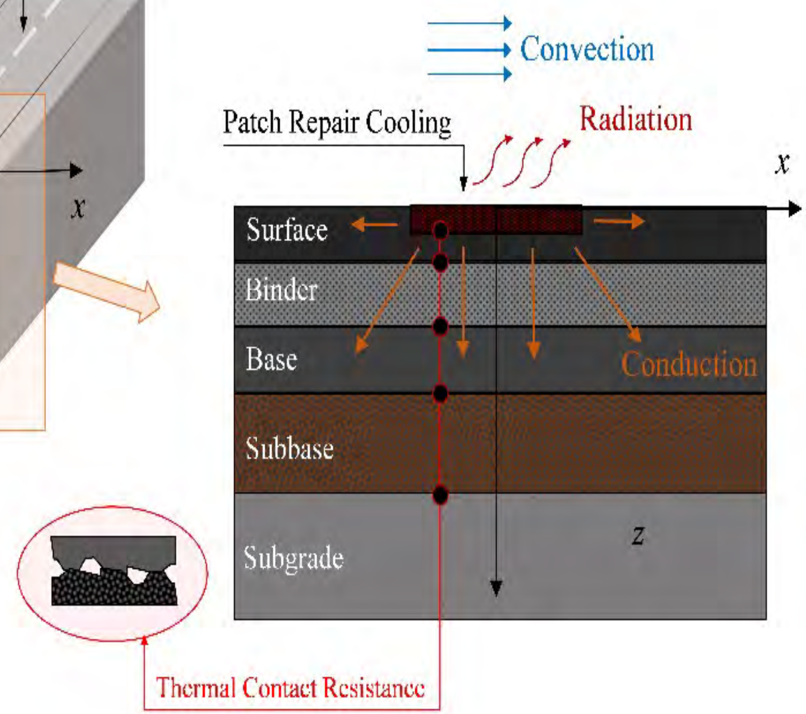
- **Temporary repairs** can be rapidly deployed and are relatively cheap but deteriorate fast and constitute short term solutions.
- **Permanent patch repairs** are preferable although they deteriorate too. The quality and durability depends on the cooling rate of the hot patch mix used. For proper bonding and repair integrity, the hot mix should be kept above a minimum temperature level, e.g. 85^{o}C for 160/220 bitumen grade (BS 594987), while compaction takes place (typically about 10 minutes).

Heat Transfer in Pavement Repair Processes

- Temperature drops significantly at the surface, lower part and lateral repair boundaries. This can cause **weak spots** and **compromise the repair integrity and durability**.

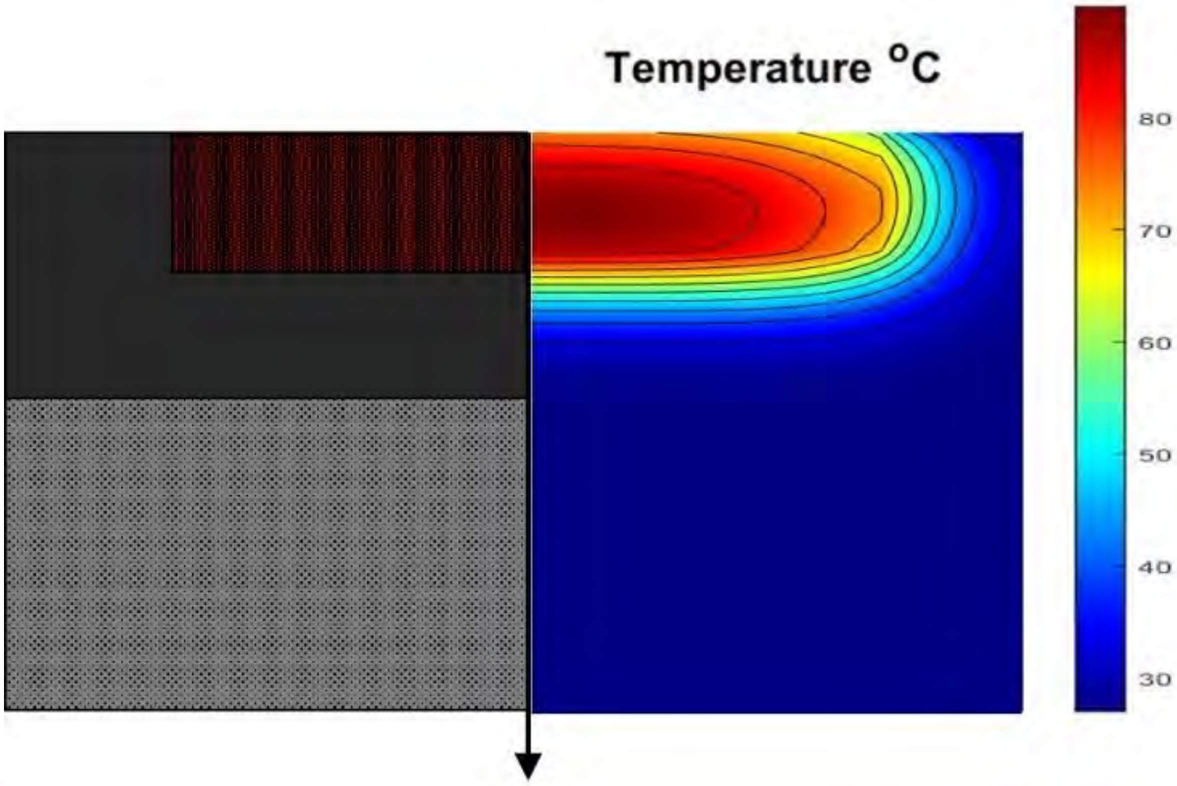


- **Heat losses** (conduction, convection, radiation)



Heat Transfer in Pavement Repair Processes

- Accurate Heat Transfer analysis is of outmost importance.
- We develop **customised Finite Element Heat Transfer solvers**
- **Aim**: Accurate and fast analysis to be used in real time control of heaters.



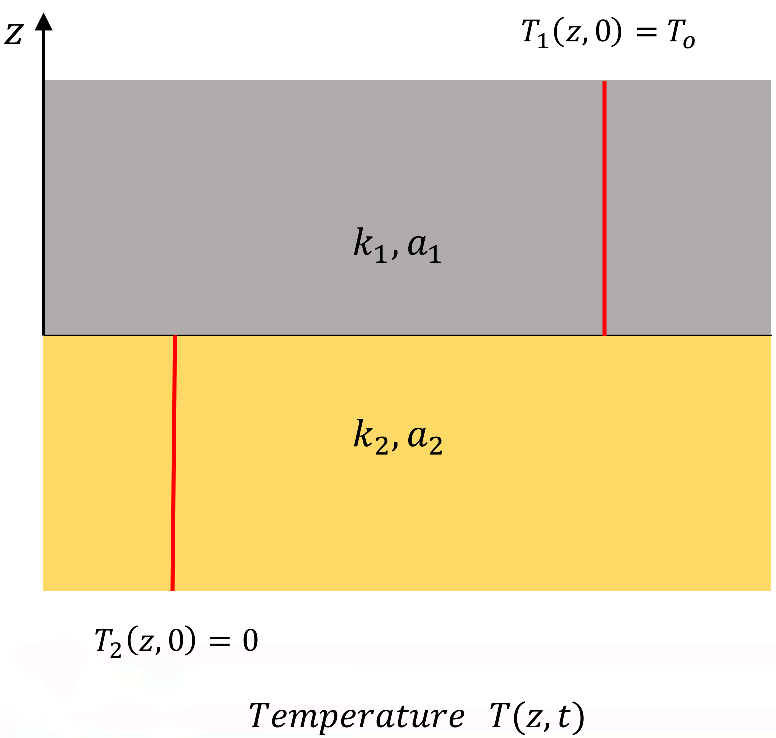
Heat Transfer in Pavement Repair Processes

- **Thermal Contact Resistance**

$$\frac{T_1}{T_o} = \frac{K}{1+K} + \frac{1}{1+K}\left[\operatorname{erf}\left(\frac{z}{2\sqrt{a_1 t}}\right) + e^{b_1 z + b_1^2 a_1 z}\operatorname{erfc}\left(\frac{z}{2\sqrt{a_1 t}} + b_1\sqrt{a_1 t}\right)\right]$$

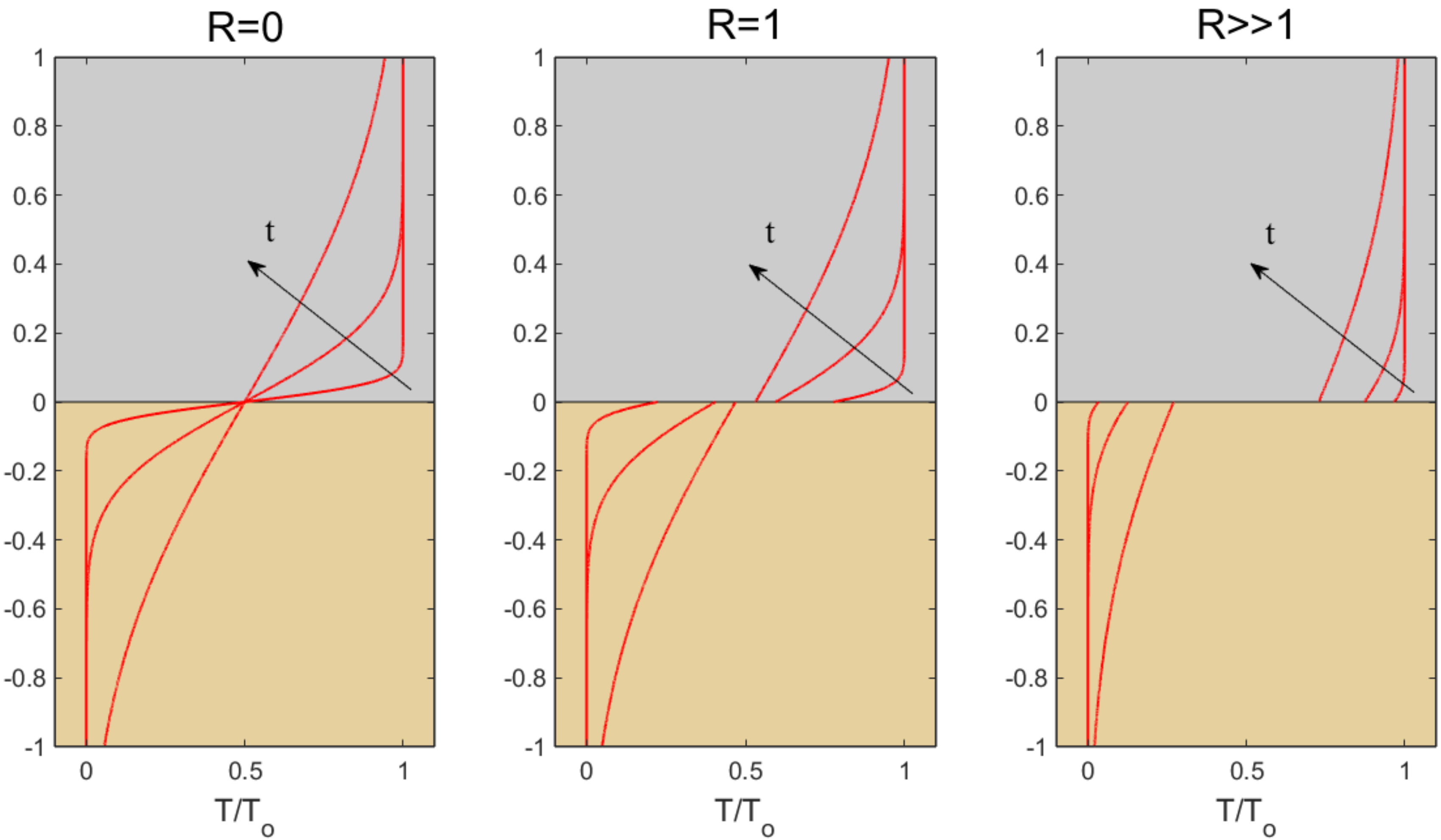
$$\frac{T_2}{T_o} = \frac{K}{1+K}\left[\operatorname{erfc}\left(\frac{|z|}{2\sqrt{a_2 t}}\right) - e^{b_2 z + b_2^2 a_2 z}\operatorname{erfc}\left(\frac{|z|}{2\sqrt{a_2 t}} + b_2\sqrt{a_2 t}\right)\right]$$

$$K = \frac{k_1}{k_2}\sqrt{\frac{a_2}{a_1}} \qquad b_1 = \frac{1+K}{Rk_1}, \quad b_2 = \frac{1+K}{KRk_2}$$

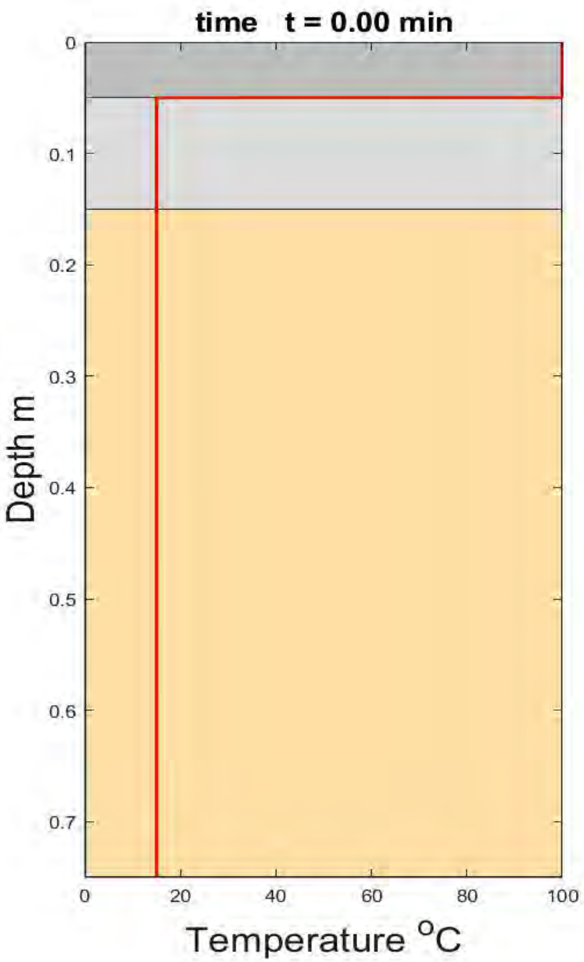


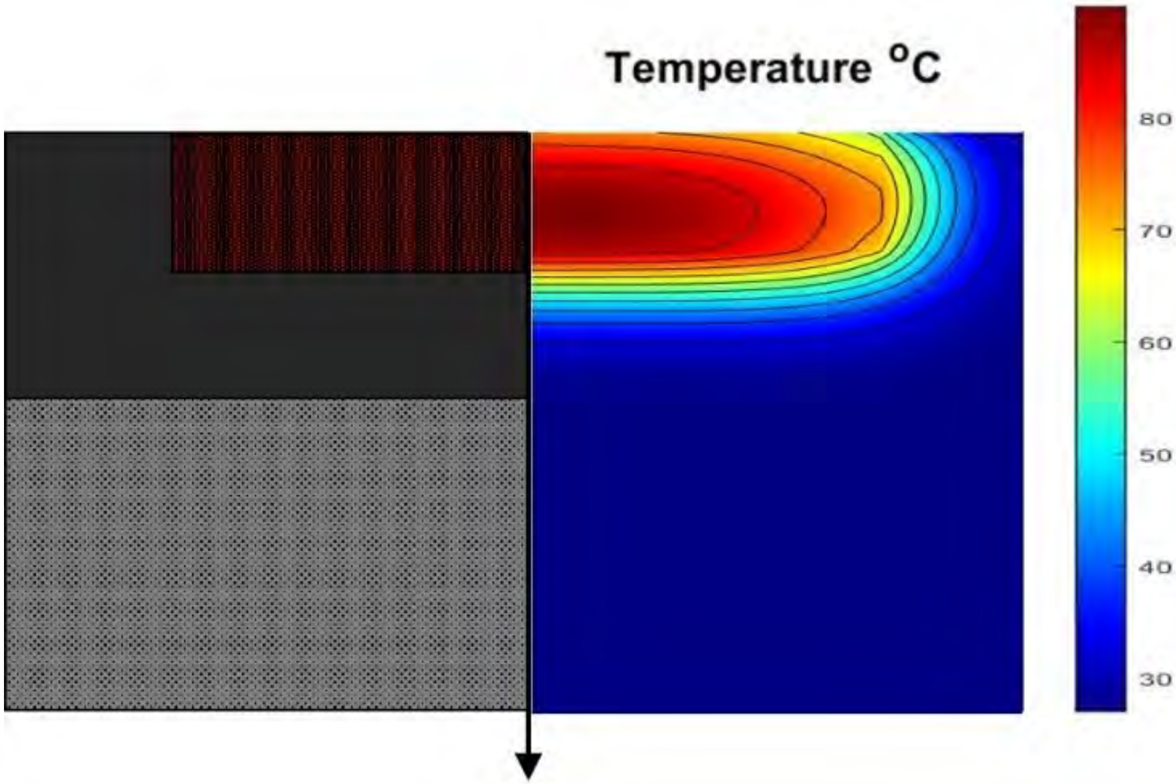
(Schaaf, *Quart. Appl. Math.* 1947)

Heat Transfer in Pavement Repair Processes



Heat Transfer in Pavement Repair Processes





CME Based FEM

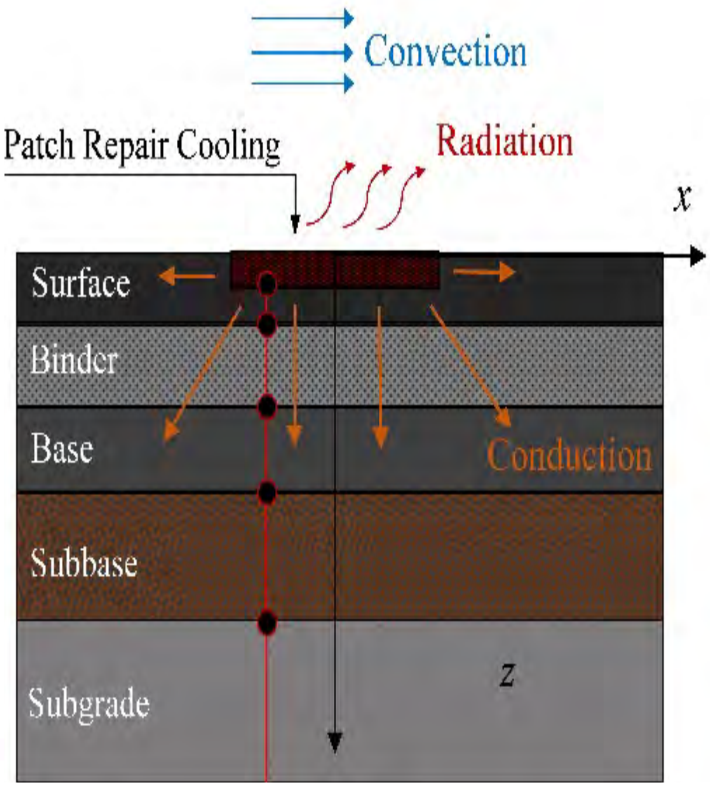
- *M Layers* $\frac{\partial T_\ell}{\partial t} - a_\ell \nabla^2 T_\ell = 0$

$$\rho C \frac{\partial T}{\partial t} - \nabla \cdot \left(k \nabla T \right) = 0$$

- *Thermal Diffusivity* $a_\ell = \frac{k_\ell}{\rho_\ell C_\ell}$
- *Convection (upper surface)* $k \frac{\partial T}{\partial z} = h(T - T_\infty)$
- *Adiabatic (lower surface)* $\frac{\partial T}{\partial z} = 0$
    - *Interfaces (thermal contact resistance)*

$$k_\ell \frac{dT_\ell}{dz} = k_{\ell+1} \frac{dT_{\ell+1}}{dz}, \quad \ell = 1, 2, \ldots, M-1$$

$$k_\ell \frac{dT_\ell}{dz} - R_{\ell+1}^{-1} \left(T_{\ell+1} - T_\ell \right) = 0, \quad \ell = 1, 2, \ldots, M-1$$



CME Based FEM

- *Variational Form*

$$\int_V \rho C w \frac{\partial T}{\partial t} dV + \int_V k \nabla w \cdot \nabla T dV - \int_A k w \nabla T \cdot \mathbf{n} dA = 0$$

- *Vertical Expansion*

$$T(x,y,z,t) = \sum_{n=1}^{N} Z_n(z) T_n(x,y,t) = \mathbf{Z}^T \mathbf{T}$$

- *Weights*

$$w = \delta T = \mathbf{Z}^T \delta \mathbf{T}$$

- *Integrate along z*

$$\int_\Omega \delta \mathbf{T}^T \mathbf{M} \mathbf{T} d\Omega + \int_\Omega \left(\nabla \delta \mathbf{T}^T \right) \mathbf{A} \nabla \mathbf{T} d\Omega + \int_\Omega \delta \mathbf{T}^T \mathbf{B} \cdot \nabla \mathbf{T} d\Omega + \int_\Omega \delta \mathbf{T}^T \mathbf{C} \mathbf{T} d\Omega = \mathbf{0}$$

CME Based FEM

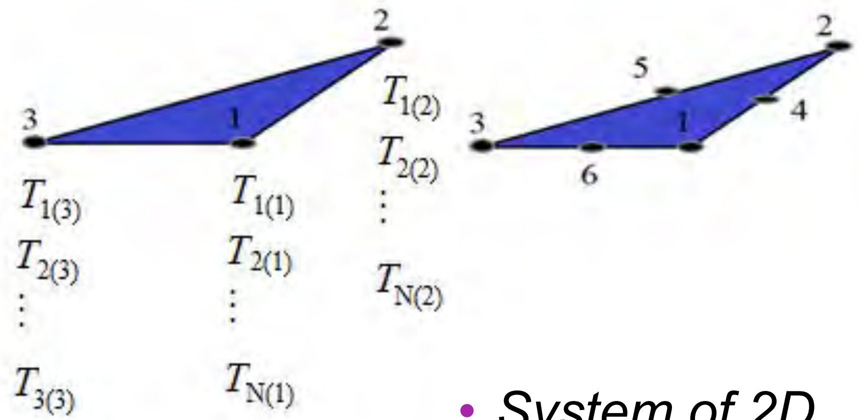
$$\int_{\Omega} \delta\mathbf{T}^T \mathbf{M}\,\mathbf{T}\,d\Omega + \int_{\Omega} \left(\nabla\delta\mathbf{T}^T\right)\mathbf{A}\nabla\mathbf{T}\,d\Omega + \int_{\Omega} \delta\mathbf{T}^T \mathbf{B}\cdot\nabla\mathbf{T}\,d\Omega + \int_{\Omega} \delta\mathbf{T}^T \mathbf{C}\,\mathbf{T}\,d\Omega = \mathbf{0}$$

$$\mathbf{M} = \sum_{\ell=1}^{M} \int_{z_\ell}^{z_{\ell+1}} \rho_\ell C_\ell Z_m Z_n dz$$

$$\mathbf{A} = \sum_{\ell=1}^{M} \int_{z_\ell}^{z_{\ell+1}} k_\ell Z_m Z_n dz$$

$$\mathbf{B} = \sum_{\ell=1}^{M} 2sym \int_{z_\ell}^{z_{\ell+1}} k_\ell \left(\frac{\partial Z_m}{\partial x} Z_n + \frac{\partial Z_m}{\partial y} Z_n \right) dz$$

$$\mathbf{C} = \sum_{\ell=1}^{M} \int_{z_\ell}^{z_{\ell+1}} k_\ell \left(\frac{\partial Z_m}{\partial x}\frac{\partial Z_n}{\partial x} + \frac{\partial Z_m}{\partial y}\frac{\partial Z_n}{\partial y} + \frac{\partial Z_m}{\partial z}\frac{\partial Z_n}{\partial z} \right) dz$$



- *System of 2D equations*

CME Based FEM

- *We need to construct the vertical basis*

$$\frac{d^2 Z_\ell}{dz^2} + \frac{\lambda^2}{a_\ell^2} Z_\ell = 0, \quad z_\ell < z < z_{\ell+1}, \quad \ell = 1, 2, \ldots, M$$

$$Z_\ell(z) = C_1 \cos\left[\lambda z / a_\ell\right] + C_2 \sin\left[\lambda z / a_\ell\right] = C_1 \Phi_1 + C_2 \Phi_2$$

- *2M equations*

$$k_\ell \frac{dZ_\ell}{dz} = k_{\ell+1} \frac{dZ_{\ell+1}}{dz}, \quad \ell = 1, 2, \ldots, M-1$$

$$k_1 \frac{dZ_1}{dz} - hZ_1 = 0$$

$$k_\ell \frac{dZ_\ell}{dz} - R_{\ell+1}^{-1}\left(Z_{\ell+1} - Z_\ell\right) = 0, \quad \ell = 1, 2, \ldots, M-1$$

$$\frac{dZ_M}{dz} = 0$$

CME Based FEM

- *Rewrite the solution within each layer in the form*

$$Z_\ell(z) = T_* L_1[\Phi_1, \Phi_2] + T_{**} L_2[\Phi_1, \Phi_2]$$

(Mikhailov & Ozisik, Unified Analysis and Solutions of Heat1and Mass Transfer, 1984)

- ***M+1 equations!!!***

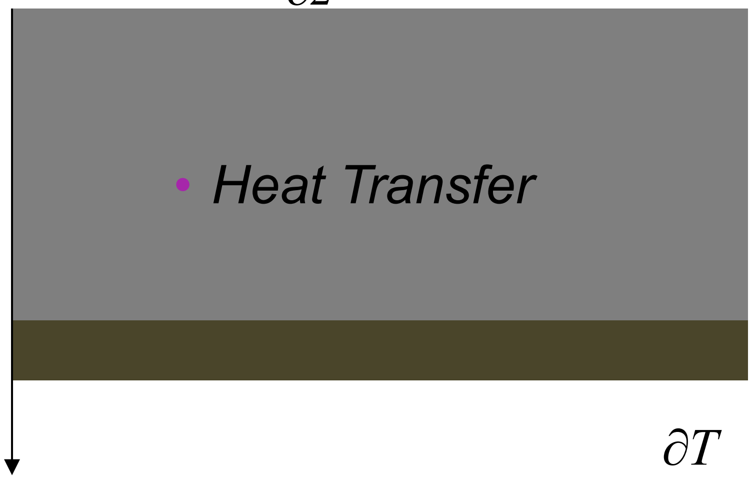
- *Solve transcendental equation*

$$\det\left[\mathbf{K}(\lambda)\right] = 0$$

$$\mathbf{K}(\lambda) = \begin{bmatrix} a_0 & b_1 & 0 & 0 & . & 0 \\ b_1 & a_2 & b_2 & 0 & . & 0 \\ 0 & b_2 & a_2 & . & . & . \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & a_{N-1} & b_N \\ 0 & 0 & . & 0 & b_N & a_N \end{bmatrix}$$

Water Waves?

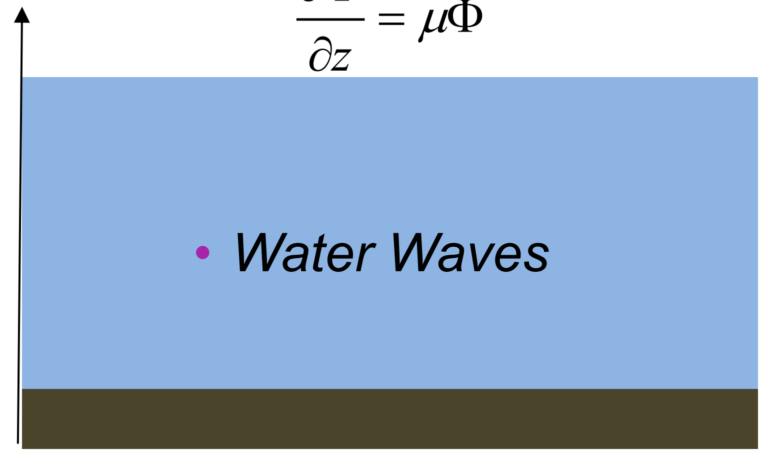
$$k\frac{\partial T}{\partial z} = h(T - T_\infty)$$



$$\frac{\partial T}{\partial z} = 0$$

$$\lambda_n H \tan(\lambda_n H) = bH$$

$$\frac{\partial \Phi}{\partial z} = \mu\Phi$$



- *Propagating mode* $\quad \frac{\partial \Phi}{\partial z} = 0$

$$\kappa_0 \tanh(\kappa_0) = \mu H$$

- *Evanescent modes*

$$\kappa_n \tan(\kappa_n) = - \mu H$$

(Chamberlain & Porter, 1999)

Water Waves?

2nd and 3rd Order Root Approximation Algorithms

Given ${}^{0}\kappa_n$, for $j = 0,1,2,\ldots$

- **2nd Order Method:**

$$ {}^{j+1}\kappa_n = n\pi + \frac{\mu\left(n\pi - {}^{j}\kappa_n\right)}{\mu^2 + {}^{j}\kappa_n^2 - \mu} - \frac{\mu^2 + {}^{j}\kappa_n^2}{\mu^2 + {}^{j}\kappa_n^2 - \mu}\operatorname{Arctan}\left(\frac{\mu}{{}^{j}\kappa_n}\right) $$

- **3rd Order Method:**

$$ {}^{j+1}\kappa_n = n\pi + \frac{\mu\left(n\pi - {}^{j}\kappa_n\right)}{\mu^2 + {}^{j}\kappa_n^2 - \mu} - \frac{\mu^2 + {}^{j}\kappa_n^2}{\mu^2 + {}^{j}\kappa_n^2 - \mu}\operatorname{Arctan}\left(\frac{\mu}{{}^{j}\kappa_n}\right) - \frac{{}^{j}\kappa_n\mu\left(\mu^2 + {}^{j}\kappa_n^2\right)}{\left(\mu^2 + {}^{j}\kappa_n^2 - \mu\right)^3}\left[n\pi - {}^{j}\kappa_n - \operatorname{Arctan}\left(\frac{\mu}{{}^{j}\kappa_n}\right)\right]^2 $$

$$ {}^{0}\kappa_n = n\pi - \left[\frac{\mu^2 + n^2\pi^2 - \mu}{\mu^2 + n^2\pi^2 - 2\mu}\right]\operatorname{Arctan}\left(\frac{\mu}{n\pi}\right) $$

(P, Papoutselis, Athanassoulis, *J. Eng. Math*. 2016)

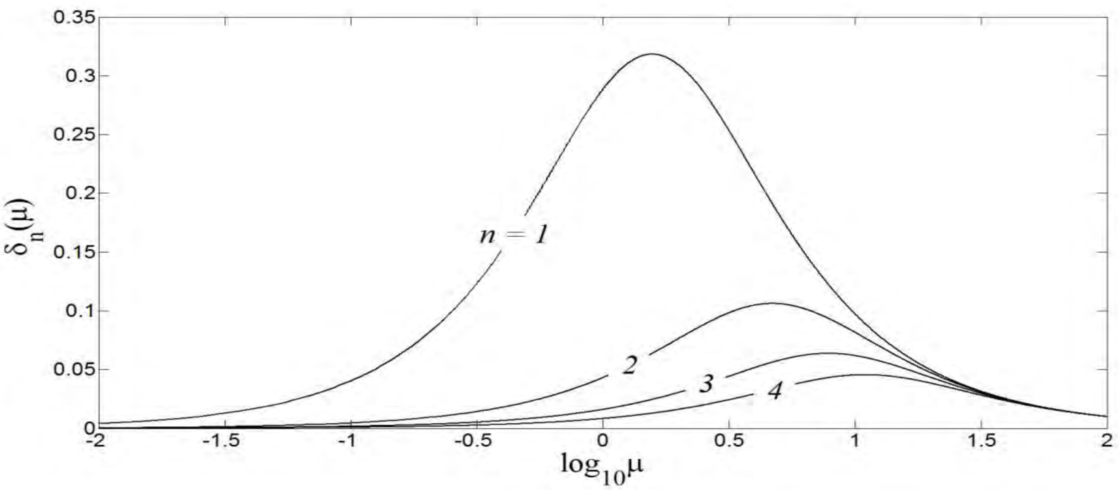
Water Waves?

- *ERROR ESTIMATES*

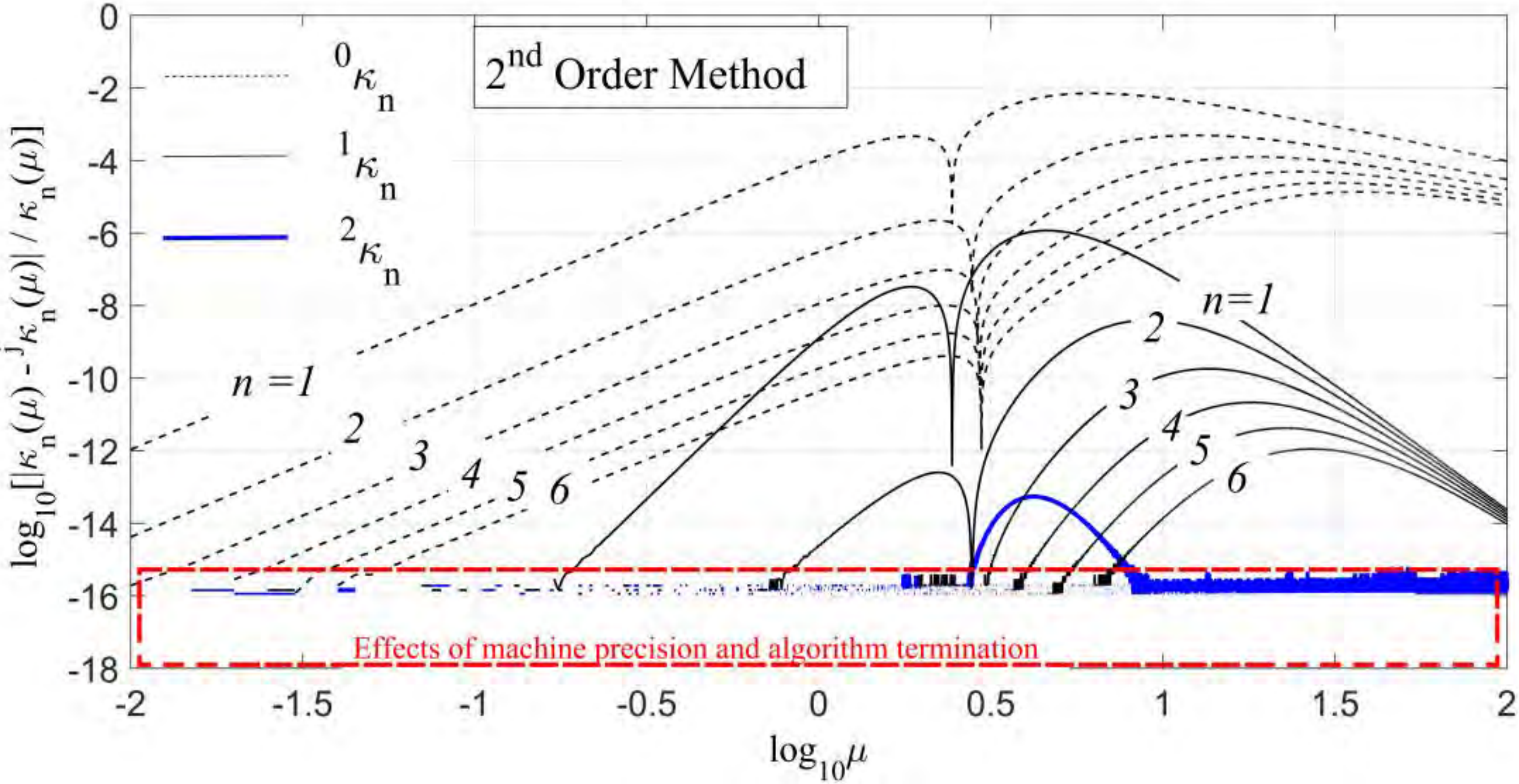
$$\left| {}^{j+1}\kappa_n - \kappa_n \right| \le r_n(\mu)\delta_n^2(\mu)\left| {}^{j}\kappa_n - \kappa_n \right|^2 + O\left(\left| {}^{j}\kappa_n - \kappa_n \right|^3\right)$$

$$\left| {}^{j+1}\kappa_n - \kappa_n \right| \le R_n(\mu)\delta_n^3(\mu)\left| {}^{j}\kappa_n - \kappa_n \right|^3 + O\left(\left| {}^{j}\kappa_n - \kappa_n \right|^4\right)$$

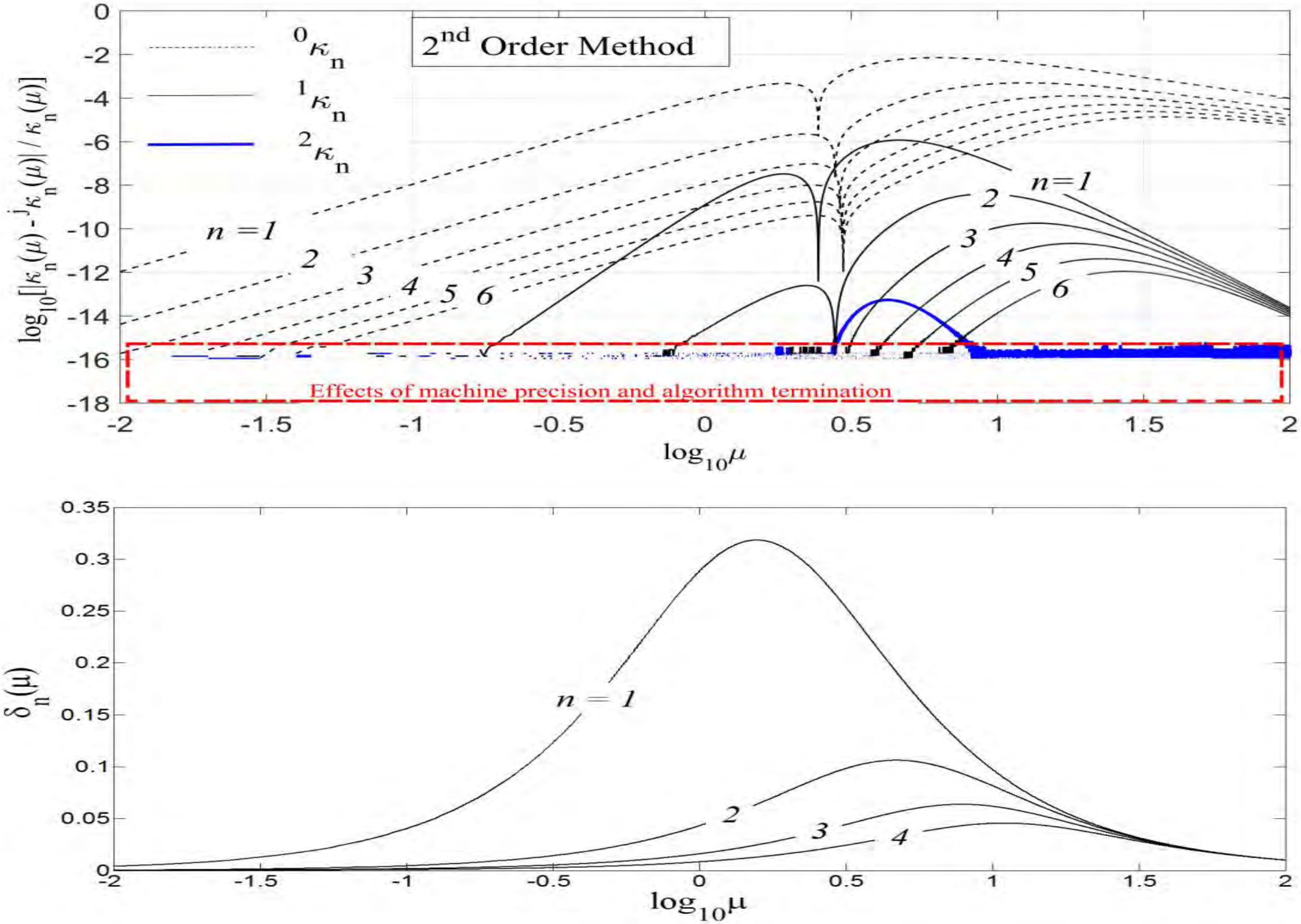
$$\delta_n(\mu) = \frac{4\mu}{(2n-1)^2\pi^2 + 4\mu^2} < 1$$



Water Waves?


2nd Order Method
$^0\kappa_n$
$^1\kappa_n$
$^2\kappa_n$
n =1
2
3
4
5
6
n=1
2
3
4
5
6
Effects of machine precision and algorithm termination
$\log_{10}[|\kappa_n(\mu) - {}^j\kappa_n(\mu)| / \kappa_n(\mu)]$
$\log_{10}\mu$
0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2

Water Waves?



2nd Order Method
$^0\kappa_n$
$^1\kappa_n$
$^2\kappa_n$
n = 1
2
3
4
5
6
Effects of machine precision and algorithm termination
$\log_{10}[|\kappa_n(\mu) - {}^j\kappa_n(\mu)| / \kappa_n(\mu)]$
$\log_{10}\mu$
$\delta_n(\mu)$

Καλή Συνέχεια κύριε Μάκη!