**National Technical University of Athens**

School of Naval Architecture and Marine Engineering

***Seas, Probabilities and Memories***

# Numerical Solution of Generalized FPK Equations in Stochastic Dynamics, Using PUFEM

**Nikolaos P. Nikoletatos-Kekatos**

# Problem under consideration

Our starting point is the initial-value problem (IVP) for a non-linear, two-dimensional system of RDEs, reading:

$$\dot{X}_n(t;\theta) = h_n\big(\boldsymbol{X}(t;\theta),t\big) + \Xi_n(t;\theta), \qquad X_n(t_0;\theta) = X_n^0(\theta), \qquad n = 1,2, \qquad (1)$$

where

the overdot in (1) denotes differentiation with respect to time,

$\theta$ denote the stochastic argument,

$h_1(\boldsymbol{x}(t),t), h_2(\boldsymbol{x}(t),t)$ are continuous, deterministic functions.

Initial value $\boldsymbol{X}^0(\theta)$ and excitation $\boldsymbol{\Xi}(\bullet,\theta)$ are considered correlated and jointly Gaussian, which constitute **the data of the system**.

- The **probabilistic structures** of the initial value and excitation, are completely defined by means of their mean vectors $\boldsymbol{m}_{\boldsymbol{X}^0}$ and $\boldsymbol{m}_{\boldsymbol{\Xi}(\bullet)}(t)$, autocovariances matrices $\boldsymbol{C}_{\boldsymbol{X}^0\boldsymbol{X}^0}$, $\boldsymbol{C}_{\boldsymbol{\Xi}(\bullet)\boldsymbol{\Xi}(\bullet)}$ and the cross-covariance matrix $\boldsymbol{C}_{\boldsymbol{X}^0\boldsymbol{\Xi}(\bullet)}$.

# Generalized FPK equation corresponding to a two-dimensional system of RDEs

- To study the probabilistic structure of the response $\boldsymbol{X}(t;\theta)$, of the system (1), equations with respect to the one-time probability density function (pdf) $f_{X_1(t)X_2(t)}(\boldsymbol{x}) = f_{\boldsymbol{X}(t)}(\boldsymbol{x})$ are formulated.

The one-time pdf-evolution equation (genFPK) (Mamis, Athanassoulis and Kapelonis, 2019) (Mamis, 2020) corresponding to the IVP (1), reads:

$$\partial_t f_{\boldsymbol{X}(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\Big[ q_n(x,t)\, f_{\boldsymbol{X}(t)}(\boldsymbol{x}) \Big] = \sum_{n_1=1}^{2}\sum_{n_2=1}^{2} \frac{\partial^2}{\partial x_{n_1}\partial x_{n_2}} \Big( \mathcal{D}_{n_1 n_2}\big[ f_{\boldsymbol{X}(\bullet)}(\bullet); x,t \big] \cdot f_{\boldsymbol{X}(t)}(\boldsymbol{x}) \Big). \quad (2a)$$

- Eq. (2) is a generalization of the classical Fokker-Planck-Kolmogorov (FPK) equation (will be presented below), which is a linear pde. To this end, Eq. (2a) is also called generalized FPK (genFPK).

# Generalized FPK equation corresponding to a two-dimensional system of RDEs

The one-time pdf-evolution equation (genFPK)

$$\partial_t f_{X(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\Big[ q_n(x,t)\, f_{X(t)}(\boldsymbol{x}) \Big] = \sum_{n_1=1}^{2} \sum_{n_2=1}^{2} \frac{\partial^2}{\partial x_{n_1} \partial x_{n_2}} \Big( \mathcal{D}_{n_1 n_2}\Big[ f_{X(\cdot)}(\bullet); x, t \Big] \cdot f_{X(t)}(\boldsymbol{x}) \Big). \quad (2a)$$

The quantities $\mathcal{D}_{n_1 n_2}\Big[ f_{X(\cdot)}(\bullet); x, t \Big]$ are called **generalized diffusion coefficients**, given by:

$$\mathcal{D}_{n_1 n_2}\Big[ f_{X(\cdot)}(\bullet); x, t \Big] = \sum_{\ell_1=1}^{2} C_{X^0_{\ell_1} \Xi_{n_1}(\cdot)}(t)\, \mathcal{B}^{X_0\,\Xi(\cdot)}_{n_2 \ell_1}\Big[ f_{X(\cdot)}(\bullet); x, t \Big] + $$
$$+ \int_{t_0}^{t} C_{\Xi_{\ell_1}(\cdot)\Xi_{n_1}(\cdot)}(t, s)\, \mathcal{B}^{\Xi(\cdot)\,\Xi(\cdot)}_{n_2 \ell_1}\Big[ f_{X(\cdot)}(\bullet); x, t, s \Big] ds. \quad (2b)$$

- The diffusion coefficients $\mathcal{D}_{n_1 n_2}[\cdots]$ are also non-locally dependent on the unknown pdf, through its time-history, from the initial time up to the current time $t$
- The diffusion coefficients $\mathcal{D}_{n_1 n_2}[\cdots]$ are non-linearly dependent on $\boldsymbol{x}$, by means of the functions $\mathcal{B}^{X_0\,\Xi(\cdot)}_{n_2 \ell_1}[\cdots]$ and $\mathcal{B}^{\Xi(\cdot)\,\Xi(\cdot)}_{n_2 \ell_1}[\cdots]$.

# Generalized FPK equation corresponding to a two-dimensional system of RDEs

In the derivation of the pdf-evolution equation (2a, b), in place of the functions $\mathcal{B}^{X_0\,\Xi(\bullet)}_{n_2\ell_1}[\cdots]$ and $\mathcal{B}^{\Xi(\bullet)\,\Xi(\bullet)}_{n_2\ell_1}[\cdots]$, the following quantities emerge:

$$\mathcal{T}^{\boldsymbol{X}^0}_{n_2\ell_1}\big[\boldsymbol{X}(\bullet);t\big] = \mathbb{E}^{\theta}\Big[\delta(\boldsymbol{x}-\boldsymbol{X}(t;\theta))\,\underline{V^{\boldsymbol{X}^0}_{n_2\ell_1}(t;\theta)}\Big], \quad V^{\boldsymbol{X}^0}_{n_2\ell_1}(t;\theta) = \frac{\partial X_{n_2}(t)}{\partial X^0_{\ell_1}}$$

$$\mathcal{T}^{\boldsymbol{\Xi}(s)}_{n_2\ell_1}\big[\boldsymbol{X}(\bullet);t\big] = \mathbb{E}^{\theta}\Big[\delta(\boldsymbol{x}-\boldsymbol{X}(t;\theta))\,\underline{V^{\boldsymbol{\Xi}(s)}_{n_2\ell_1}(t;\theta)}\Big], \quad V^{\boldsymbol{\Xi}(s)}_{n_2\ell_1}(t;\theta) = \frac{\delta X_{n_2}(t)}{\delta \Xi_{\ell_1}(s)}$$

Variational Derivatives

- Using the initial system of RDEs, homogeneous linear systems of ODEs are formulated with respect to the quantities $V^{\boldsymbol{X}^0}_{n_2\ell_1}$, $V^{\boldsymbol{\Xi}(s)}_{n_2\ell_1}$, giving the expressions:

$$V^{\boldsymbol{X}^0}_{n_2\ell_1}(t;\theta) = \Phi_{n_2\ell_1}\Big[\boldsymbol{J}\Big(\boldsymbol{X}(\bullet|_{t_0}^{t};\theta),\bullet|_{t_0}^{t}\Big)\Big], \quad V^{\boldsymbol{\Xi}(s)}_{n_2\ell_1}(t;\theta) = \Phi_{n_2\ell_1}\Big[\boldsymbol{J}\Big(\boldsymbol{X}(\bullet|_{s}^{t};\theta),\bullet|_{s}^{t}\Big)\Big],$$

where $\boldsymbol{J}$ is the Jacobian matrix and $\Phi$ denotes the state-transition matrix.

# Generalized FPK equation corresponding to a two-dimensional system of RDEs

❖ Decomposing the Jacobian matrix $\boldsymbol{J}$ in two parts:

$$\boldsymbol{J}(\cdots) = \boldsymbol{R}[\cdots] + \boldsymbol{\Delta}(\cdots),$$

where $\boldsymbol{R}[\cdots] = \mathbb{E}^{\theta}\left[\boldsymbol{J}(\cdots)\right]$ (mean value) and $\boldsymbol{\Delta}(\cdots) = \boldsymbol{J}(\cdots) - \mathbb{E}^{\theta}\left[\boldsymbol{J}(\cdots)\right]$ (fluctuations), we obtain:

$$\boldsymbol{\Phi}\left[\boldsymbol{J}(\cdots)\right] = \boldsymbol{\Phi}\left[\boldsymbol{R}[\cdots]\right]\boldsymbol{\Phi}\left[\boldsymbol{B}(\cdots)\right], \quad \boldsymbol{B}(\cdots) = \boldsymbol{\Phi}^{-1}\left[\boldsymbol{R}[\cdots]\right]\boldsymbol{\Delta}(\cdots)\,\boldsymbol{\Phi}\left[\boldsymbol{R}[\cdots]\right].$$

Introducing appropriate current-time approximations for the matrices $\boldsymbol{\Phi}\left[\boldsymbol{B}(\cdots)\right]$:

$$\mathcal{B}_{n_2\ell_1}^{X_0\,\Xi(\bullet)}\left[f_{X(\bullet)}(\bullet);x,t\right] = \left(\exp\left(\boldsymbol{\Delta}(\boldsymbol{x},t)(t-t_0)\right)\boldsymbol{\Phi}\left[\boldsymbol{R}\left[f_{X(\bullet|_{t_0}^{t})}(\bullet),\bullet|_{t_0}^{t}\right]\right]\right)_{n_2\ell_1}, \qquad (3a)$$

$$\mathcal{B}_{n_2\ell_1}^{\Xi(\bullet)\,\Xi(\bullet)}\left[f_{X(\bullet)}(\bullet);x,t,s\right] = \left(\exp\left(\boldsymbol{\Delta}(\boldsymbol{x},t)(t-s)\right)\boldsymbol{\Phi}\left[\boldsymbol{R}\left[f_{X(\bullet|_{s}^{t})}(\bullet),\bullet|_{s}^{t}\right]\right]\right)_{n_2\ell_1}. \qquad (3b)$$

❖ $\boldsymbol{\Phi}\left[\boldsymbol{R}[\cdots]\right]$ contains moments of the pdf up to the current time $t$. The unknown moments at the current time are determined numerically by extrapolation, using iterations.

# Classical FPK equation corresponding to a two-dimensional system of RDEs

Under the assumption of zero-mean, **delta-correlated Gaussian** excitation (**Gaussian white noise**):

$$\boldsymbol{C}^{\mathrm{WN}}_{\Xi(\cdot)\Xi(\cdot)}(t,s) = 2\boldsymbol{D}(t)\delta(t-s), \quad \text{where} \quad \boldsymbol{D}(\bullet) \text{ is the noise intensity matrix,}$$

the classical Fokker-Planck-Kolmogorov equation (FPK) (Pugachev and Sinitsyn, 2002) corresponding to the system (1) reads:

$$\partial_t f_{\boldsymbol{X}(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\left[ h_n(\boldsymbol{x}) f_{\boldsymbol{X}(t)}(\boldsymbol{x})\right] = \sum_{n_1=1}^{2}\sum_{n_2=1}^{2} \mathcal{D}^{\mathrm{WN}}_{n_1 n_2}(t) \frac{\partial^2 f_{\boldsymbol{X}(t)}(\boldsymbol{x})}{\partial x_{n_1} \partial x_{n_2}}, \tag{4}$$

where the diffusion coefficients are given by:

$$\mathcal{D}^{\mathrm{WN}}_{n_1 n_2}(t) = \sum_{\ell_1=1}^{2} \delta_{n_2 \ell_1} D_{\ell_1 n_1}(t),$$

# Stochastic oscillators

All **additively excited stochastic oscillators** of the form

$$\ddot{X}(t;\theta) + b\dot{X}(t;\theta) + \eta_1 X(t;\theta) + g(X(t;\theta)) = \Xi(t;\theta), \tag{5}$$

where $g(\bullet)$ is considered a continuous, non-linear (in general) function, attain the following **state-space representation**

$$\dot{X}_1(t;\theta) = X_2(t;\theta), \tag{6a}$$

$$\dot{X}_2(t;\theta) = -bX_2(t;\theta) - \eta_1 X_1(t;\theta) - g(X_1(t;\theta)) + \Xi(t;\theta). \tag{6b}$$

The corresponding pdf-evolution equation to the above oscillators, reads:

$$\begin{aligned} &\partial_t f_{X(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\left[h_n(\boldsymbol{x}) f_{X(t)}(\boldsymbol{x})\right] + m_{\Xi_2(\cdot)}(t) \frac{\partial f_{X(t)}(\boldsymbol{x})}{\partial x_2} = \\ &= \mathcal{D}_{21}\left[ f_{X(\cdot)}(\bullet); t \right] \frac{\partial^2 f_{X(t)}(\boldsymbol{x})}{\partial x_1 \partial x_2} + \mathcal{D}_{22}\left[ f_{X(\cdot)}(\bullet); x_1, t \right] \frac{\partial^2 f_{X(t)}(\boldsymbol{x})}{\partial x_2^2}. \end{aligned} \tag{7}$$

# Stochastic linear oscillator

The one-time response pdf-evolution equation (genFPK-2D), corresponding to the linear oscillator, reads

$$\partial_t f_{X(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\left[\left( h_n(\boldsymbol{x},t) + m_{\Xi_n}(t)\right) f_{X(t)}(\boldsymbol{x})\right] = \sum_{n_2=1}^{2} \mathcal{D}_{2n_2}(t) \frac{\partial^2 f_{X(t)}(\boldsymbol{x})}{\partial x_2\, \partial x_{n_2}},$$

where

$$h_1(\boldsymbol{x},t) = x_2, \qquad h_2(\boldsymbol{x},t) = -\eta_1 x_1 - b x_2,$$

and the diffusion coefficients $\mathcal{D}_{2n_2}(t)$, have closed, linear and local forms.

- This equation despite that it is linear, maintains the non-symmetric character of the second order differential operator of the non-linear pdf-evolution equations. To this end, is a good benchmark problem for numerical methods.

# Partition of Unity Finite Element Method

- Partition of Unity Finite Element Method (PUFEM), was introduced by I. Babuška and J. M. Melenk in 1996 (Melenk and Babuška, 1996; 1997), (Babuška, Banerjee, and Osborn 2003), (Oh, Kim, and Hong 2008).

❖PUFEM was chosen for the numerical solution of the pdf-evolution equations, due to the following properties

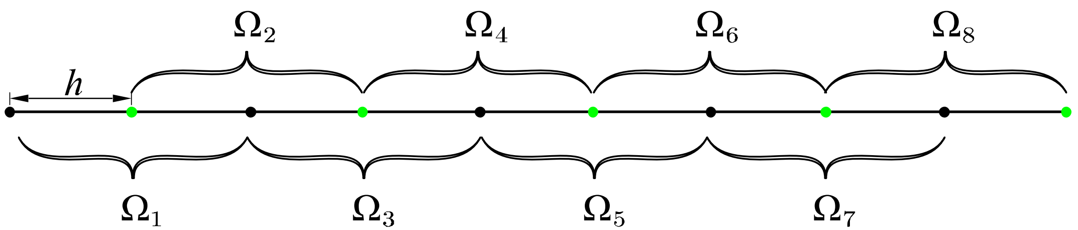
➢ It is a meshless method, avoiding the complicated meshing process of FEM, especially in the multidimensional set up.

➢ The method resolves the problem of interelement conformity (smoothness) –at **any order** and in **any space dimension** (but the curse of dimensionality remains and needs special treatment).

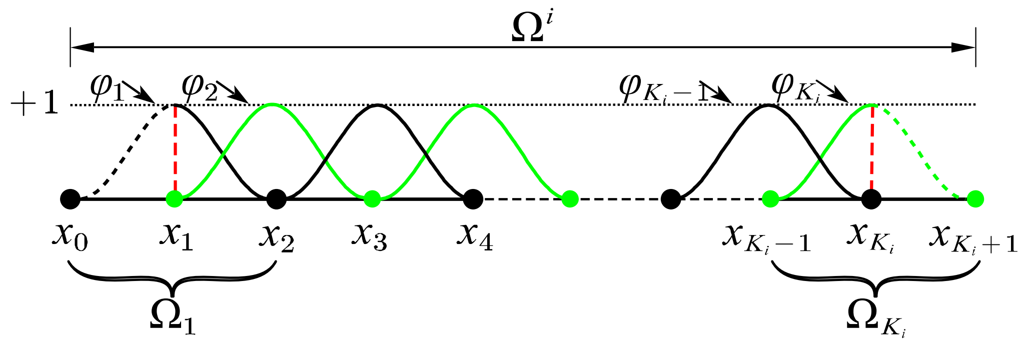
➢ PUFEM is essentially a generalization of the *h*, *p* and *hp* versions of the classical FEM

# PUFEM construction in 1D

➢ A uniform cover (patches of equal lengths) of $\Omega \subset \mathbb{R}$, $\left(\Omega_k\right)_{k=1}^{K}$, has the following layout:




➢ A partition of unity family of functions $\left(\varphi_k(\bullet) \in C^s(\Omega)\right)_k^K$ ($\boldsymbol{C^s}$**-PU**), which subordinates to the cover $\left(\Omega_k\right)_{k=1}^{K}$ of $\Omega$, has the following layout:



# PUFEM construction in 1D

➢ A local basis (LB) on each patch $\Omega_k$, $\left\{ b_\mu^k(\bullet) \in C^\ell\left(\Omega_k \longrightarrow \mathbb{R}\right),\ \mu = 1,2,\ldots,\mathrm{M}(k) \right\}$.

- In our construction we use Legendre polynomials.

❖ An approximate basis in the global domain $\Omega$ is constructed by means of the ***shape functions*** (SF): $u_\mu^k(x) = \varphi_k(x)\, b_\mu^k(x),\ x \in \Omega_k,\ \mu = 1(1)\mathrm{M}(k)$.

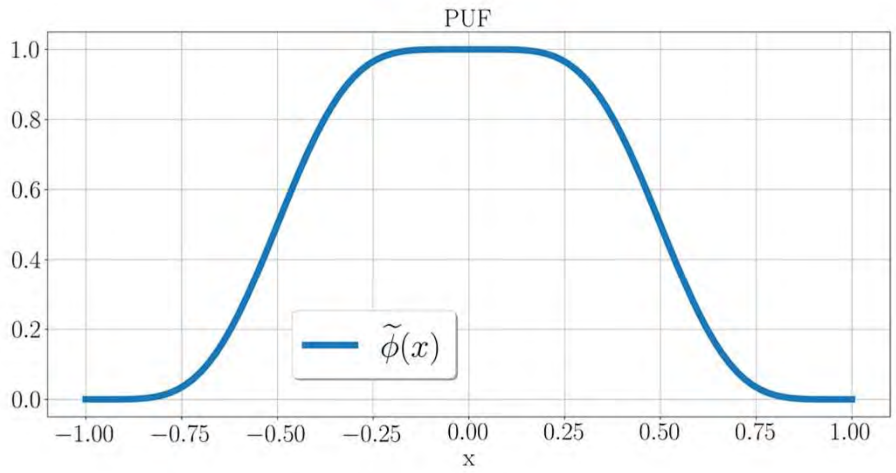



**Figure:** Reference PU function $\tilde{\varphi}_k(x)$

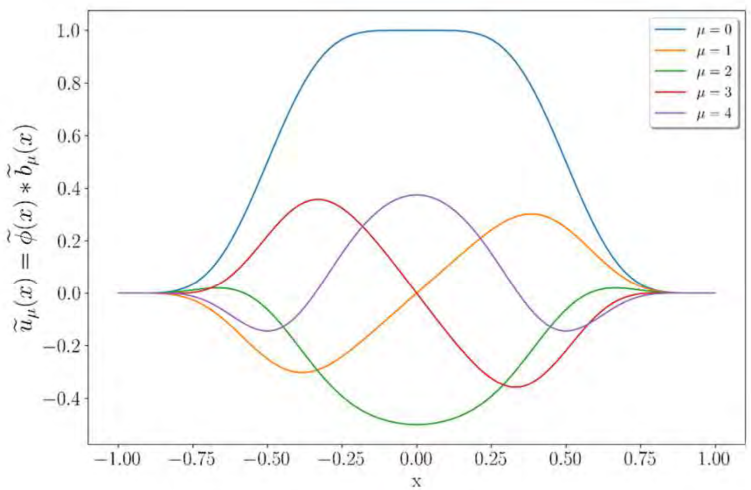



**Figure:** Reference PU-shape functions

# PUFEM construction in 2D

- A uniform 2D-cover of the **global domain** $\Omega^1 \times \Omega^2 \subset \mathbb{R} \times \mathbb{R}$, $\left(\Omega_{k_1 k_2}\right)_{k_1, k_2 = 1}^{K_1, K_2}$, is obtained by the cartesian product of two 1D-covers covering $\Omega^1$, $\Omega^2$.
- A partition of unity family which subordinates to the 2D-cover, is defined, by means of the tensor product of two 1D $C^s$ −PU families:

$$\varphi_{k_1 k_2}(x_1, x_2) = \varphi_{k_1}(x_1)\varphi_{k_2}(x_2), \quad (x_1, x_2) \in (\Omega^1_{k_1} \times \Omega^2_{k_2}).$$

- A local basis on each $\Omega_k := \Omega_{k_1} \times \Omega_{k_2}$, by means of the tensor product of two local basis sets defined on $\Omega_{k_1}$, $\Omega_{k_2}$:

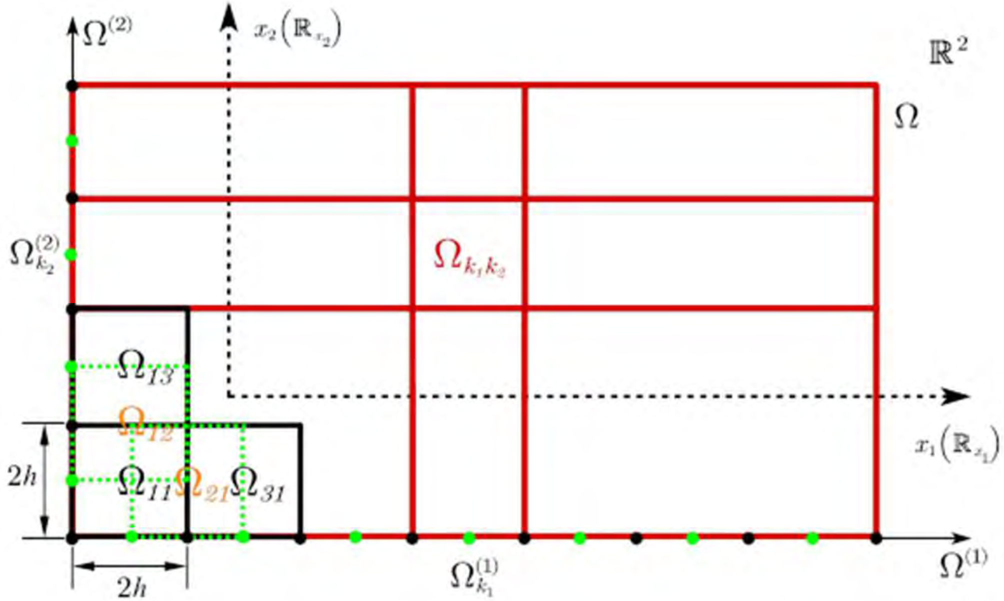
$$b^{k_1 k_2}_{\mu_1, \mu_2}(x_1, x_2) = b^{k_1}_{\mu_1}(x_1)\, b^{k_2}_{\mu_2}(x_2), \quad (x_1, x_2) \in \Omega_{k_1, k_2}, \ \mu_1 = 1(1)\mathrm{M}_1(k_1), \ \mu_2 = 1(1)\mathrm{M}_2(k_2).$$

- An approximate basis in the global domain $\Omega$ is constructed by means of the ***shape functions*** (SF), given, for $(x_1, x_2) \in \Omega_{k_1, k_2}$, by:

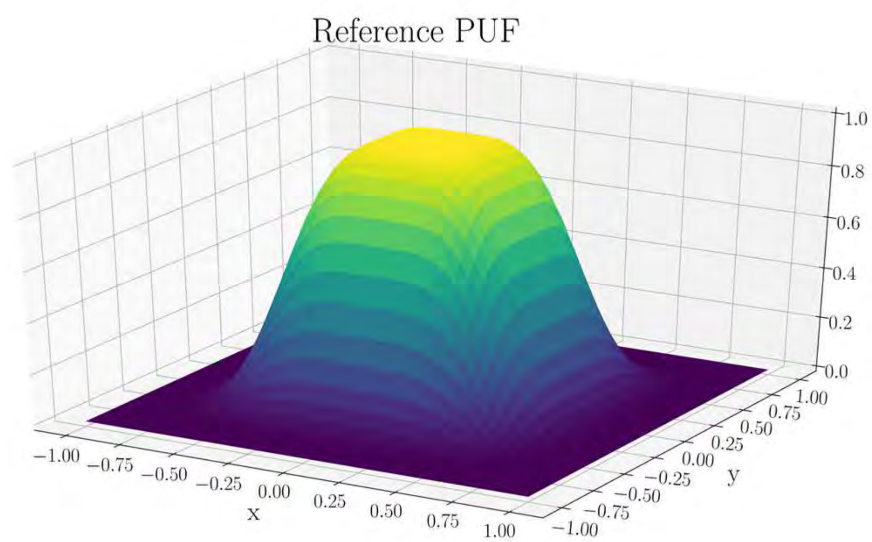
$$u^{k_1 k_2}_{\mu_1, \mu_2}(x_1, x_2) = \varphi_{k_1, k_2}(x_1, x_2)\, b^{k_1 k_2}_{\mu_1, \mu_2}(x_1, x_2), \mu_1 = 1(1)\mathrm{M}_1(k_1), \ \mu_2 = 1(1)\mathrm{M}_2(k_2).$$

# PUFEM construction in 2D

- A uniform cover of the **global domain**, $\left(\Omega_{k_1 k_2}\right)_{k_1,k_2=1}^{K_1,K_2}$, has the following layout:




- The reference 1D-PUF $\tilde{\varphi}(\xi)$ is constructed by means of a polynomial function.



# Approximate PU-Representation

❖ The global approximation space $V^{PU}(\Omega)$ by Theorem 2.1 in (Melenk & Babuška, 1996), is dense in $C(\Omega), C^1(\Omega)$ and $H^1(\Omega)$.

➢ The approximate global representation of a function $f \in C(\Omega)$, in $V^{PU}(\Omega)$, reads:

$$f(x_1, x_2) = \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} \sum_{\mu_1=1}^{M_1(k_1)} \sum_{\mu_2=1}^{M_2(k_2)} w_{\mu_1,\mu_2}^{k_1 k_2} \underset{\substack{\uparrow \\ \text{2D-SFs}}}{u_{\mu_1,\mu_2}^{k_1 k_2}(x_1, x_2)} =$$

$$= \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} \sum_{\mu_1=1}^{M_1(k_1)} \sum_{\mu_2=1}^{M_2(k_2)} w_{\mu_1,\mu_2}^{k_1 k_2} \underset{\substack{\uparrow \\ \text{1D-SFs in } k_1 \text{ patch}}}{u_{\mu_1}^{k_1}(x_1)} \underset{\substack{\uparrow \\ \text{1D-SFs in } k_2 \text{ patch}}}{u_{\mu_2}^{k_2}(x_2)} =$$

$$= \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} \sum_{\mu_1=1}^{M_1(k_1)} \sum_{\mu_2=1}^{M_2(k_2)} w_{\mu_1,\mu_2}^{k_1 k_2} \Big( \underset{\substack{\uparrow \\ \text{1D-PUF in } k_1 \text{ patch}}}{\varphi_{k_1}(x_1)} \underset{\substack{\uparrow \\ \text{1D-BFs in } k_1 \text{ patch}}}{u_{\mu_1}^{k_1}(x_1)} \Big) \left( \varphi_{k_2}(x_2)\, u_{\mu_2}^{k_2}(x_2) \right)$$

# The linear oscillator

The equation of the linear oscillator reads

$$\ddot{X}(t;\theta) + 2\zeta\omega_0 \dot{X}(t;\theta) + \omega_0^2 X(t;\theta) = \frac{\Xi(t;\theta)}{m}$$

$$\dot{X}(t_0;\theta) = \dot{X}_0(\theta), \quad X(t_0;\theta) = X_0(\theta).$$

We specify the parameters of the oscillator as

$$\zeta = 0.5, \quad \omega_0 = 1 \quad \text{and} \quad m = 1,$$

corresponding to an underdamped oscillator.

As excitation $\Xi(t;\theta)$ we consider a nonzero-mean Ornstein-Uhlembeck (OU) process, with autocorrelation function

$$C_{\Xi(\bullet)\Xi(\bullet)}(t,s) = \frac{D_{\mathrm{OU}}}{\tau_{\mathrm{cor}}} \exp\left(-\frac{|t-s|}{\tau_{\mathrm{cor}}}\right), \quad m_{\Xi} = 0.5$$

where $D_{\mathrm{OU}}$ denotes the intensity of the noise, and $\tau_{\mathrm{cor}}$ the correlation time.

# Corresponding pdf-evolution equation to the linear oscillator

The one-time response pdf-evolution equation (genFPK-2D), corresponding to the oscillator, reads

$$\partial_t f_{\boldsymbol{X}(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\Big[\Big(h_n(\boldsymbol{x},t) + m_{\Xi_n}(t)\Big) f_{\boldsymbol{X}(t)}(\boldsymbol{x})\Big] = \sum_{n_2=1}^{2} \mathcal{D}_{2n_2}(t) \frac{\partial^2 f_{\boldsymbol{X}(t)}(\boldsymbol{x})}{\partial x_2\, \partial x_{n_2}},$$

where

$$h_1(\boldsymbol{x},t) = x_2, \qquad h_2(\boldsymbol{x},t) = -\omega_0^2 x_1 - 2\zeta\omega_0 x_2,$$

and the diffusion coefficients $\mathcal{D}_{2n_2}(t)$, setting $a = -\zeta\omega_0$ and $b = \omega_0(1-\zeta^2)^{1/2}$, are expressed as

$$\mathcal{D}_{21}(t) = \int_{t_0}^{t} \boldsymbol{C}_{\Xi(\bullet)\Xi(\bullet)}(t,s) \frac{e^{a(t-s)}}{b} \sin(b(t-s))\, ds,$$

$$\mathcal{D}_{22}(t) = \int_{t_0}^{t} \boldsymbol{C}_{\Xi(\bullet)\Xi(\bullet)}(t,s) \frac{e^{a(t-s)}}{b} \big(a\sin(b(t-s)) + b\cos(b(t-s))\big)\, ds.$$

# Linear oscillator: Weak formulation

*The approximation problem reads*: For each $t \in [0, T_f]$, find $f \in V^{\mathrm{PU}}$ such that: $\forall g \in V^{\mathrm{PU}}$

$$\int_\Omega \frac{\partial f(\boldsymbol{x},t)}{\partial t} g(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{n=1}^{2} \int_\Omega [(h_n(\boldsymbol{x},t) + m_{\Xi_n}(t)) f(\boldsymbol{x},t)] \frac{\partial g(\boldsymbol{x})}{\partial x_n}\, d\boldsymbol{x} + $$

$$- \int_\Omega \mathcal{D}_{21}(t) \frac{\partial f(\boldsymbol{x},t)}{\partial x_1} \frac{\partial g(\boldsymbol{x})}{\partial x_2} + \mathcal{D}_{22}(t) \frac{\partial f(\boldsymbol{x},t)}{\partial x_2} \frac{\partial g(\boldsymbol{x})}{\partial x_2}\, d\boldsymbol{x},$$

and

$$\int_\Omega f(\boldsymbol{x},0) g(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega f_0(\boldsymbol{x}) g(\boldsymbol{x})\, d\boldsymbol{x}.$$

boundary integrals are eliminated due to the partition of unity structure

- Since the unknown pdf $f(\boldsymbol{x},t)$, is defined on $\mathbb{R}^2$, the problem is free of boundary conditions, with the understanding that the finite global domain $\Omega$ is considered, such that:

$$\int_\Omega f\, d\Omega \simeq 1.$$

# System of Equations

- Following a Bubnov-Galerkin approach in $V^{PU}(\Omega)$, the weak problem results in a linear system of the form:

$$\boldsymbol{A}\,\dot{\boldsymbol{w}}(t) = \boldsymbol{B}(t)\,\boldsymbol{w}(t)$$

- The time discretization of the problem is conducted by approximating the time derivative via a Crank-Nicolson scheme. The final system reads:

$$\left(\boldsymbol{A} - \frac{\Delta t}{2}\boldsymbol{B}(t+\Delta\tau)\right)\boldsymbol{w}(t+\Delta\tau) = \left(\frac{\Delta t}{2}\boldsymbol{B}(t) + \boldsymbol{A}\right)\boldsymbol{w}(t),$$

where $\Delta\tau$ is the time-step.

- Initialization of the numerical scheme requires to fit the PU-representation to the known initial density $f_0(\boldsymbol{x})$, obtaining the weights $\boldsymbol{w}_0 = \boldsymbol{w}(t_0)$.

# Initial fitting

Initial pdf, $f_{X(0)}(x)$, is taken Gaussian with mean value vector $m_0 = (0.25, 0.25)$ and covariance matrix $C_{X^0X^0} = \begin{bmatrix} 0.35 & 0 \\ 0 & 0.35 \end{bmatrix}$.

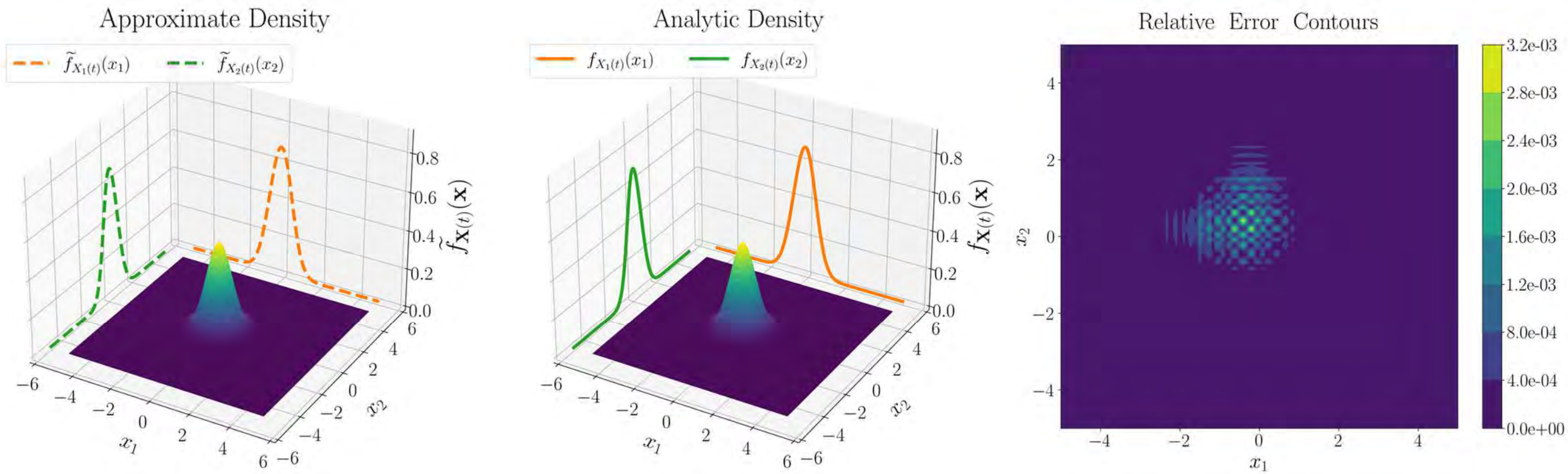



**Figure A:** Given initial pdf $f_{X(0)}(x)$ at $t = 0$

# Steady state solution

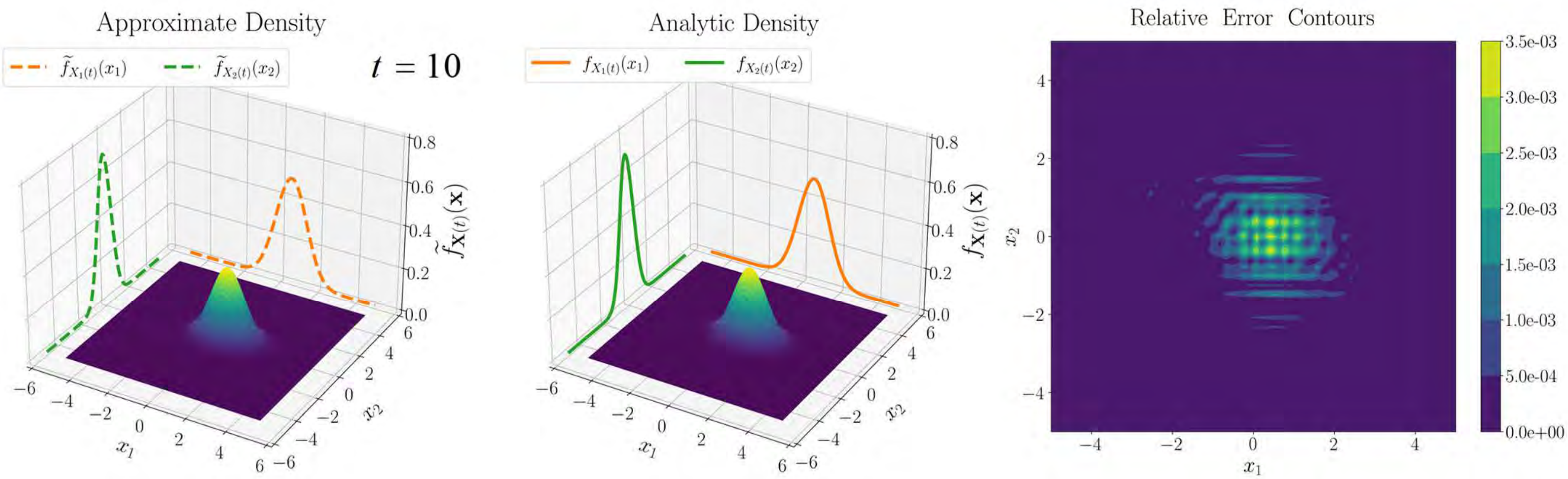
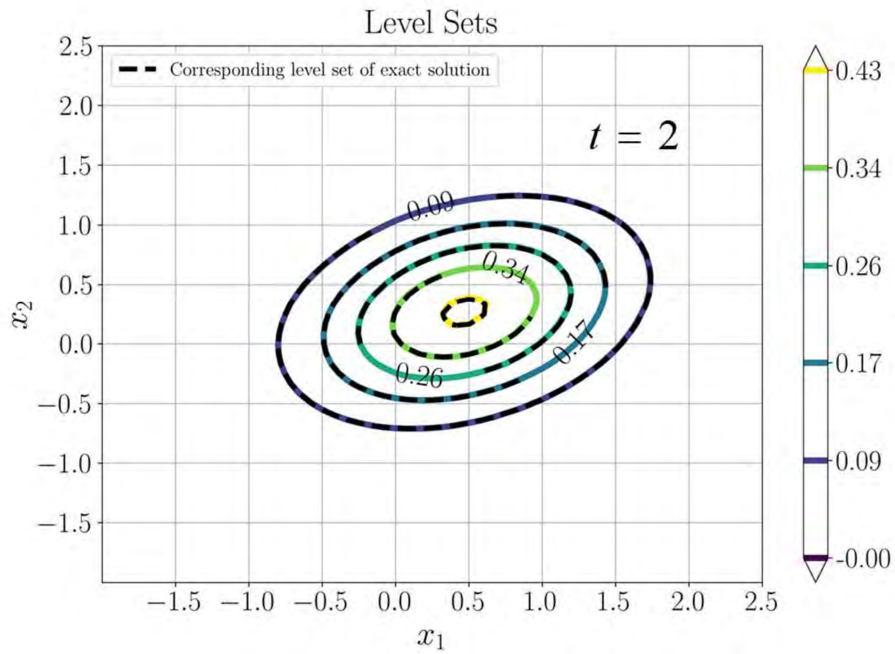
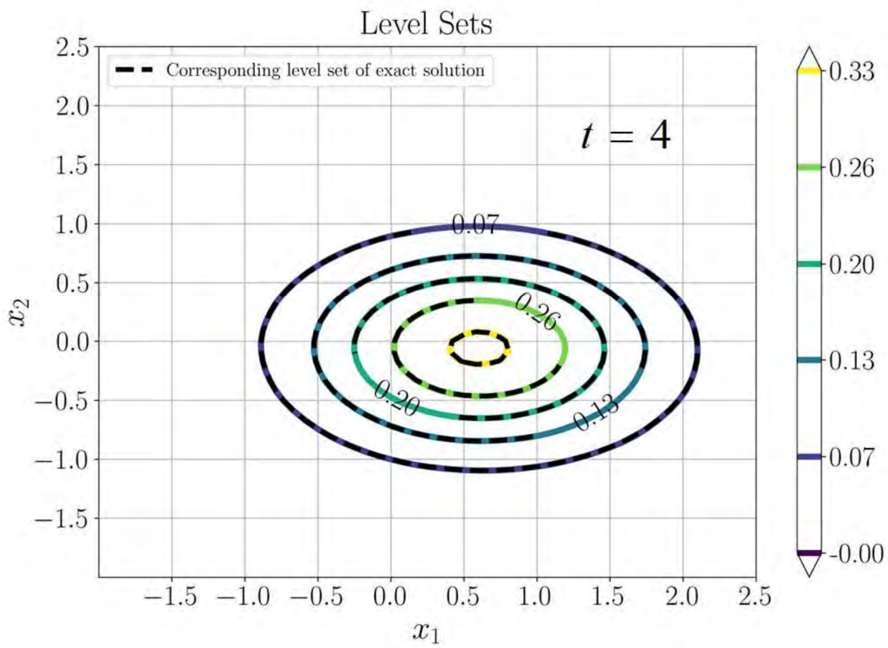


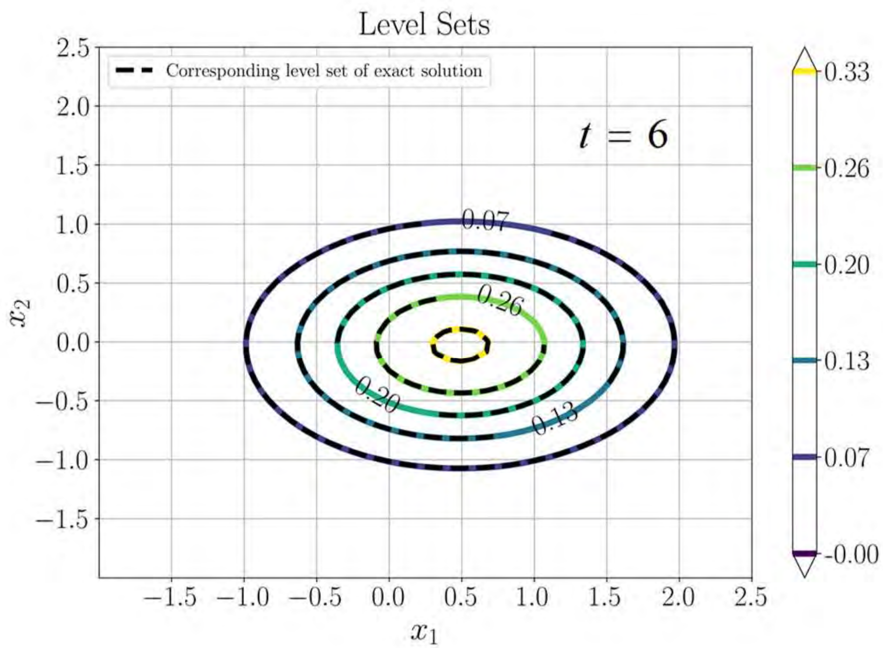

**Figure 1:** The pdf of the linear oscillator at steady state for $\tau_{cor} = 1$

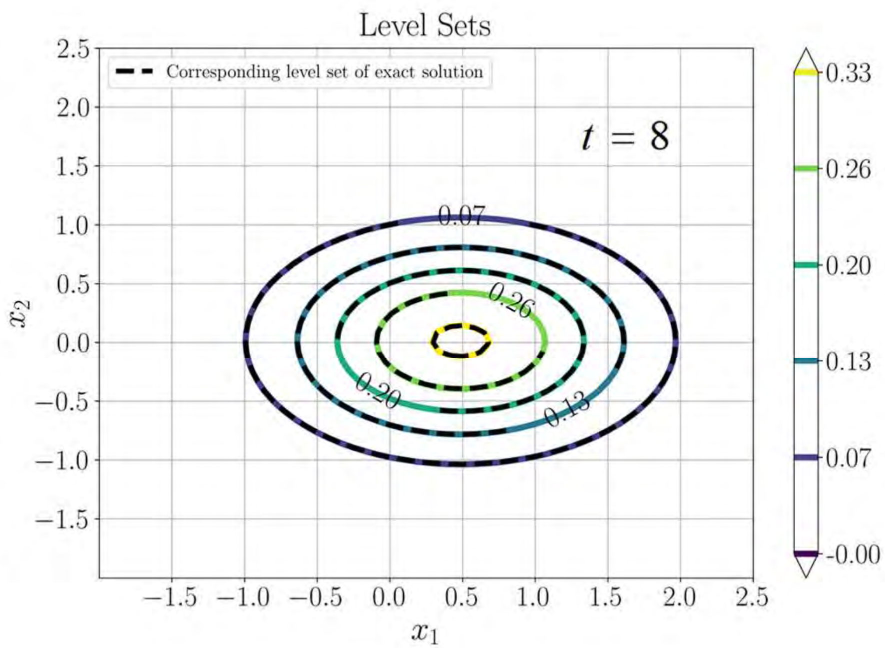
# Pdf evolution



# Pdf evolution



# Pdf evolution

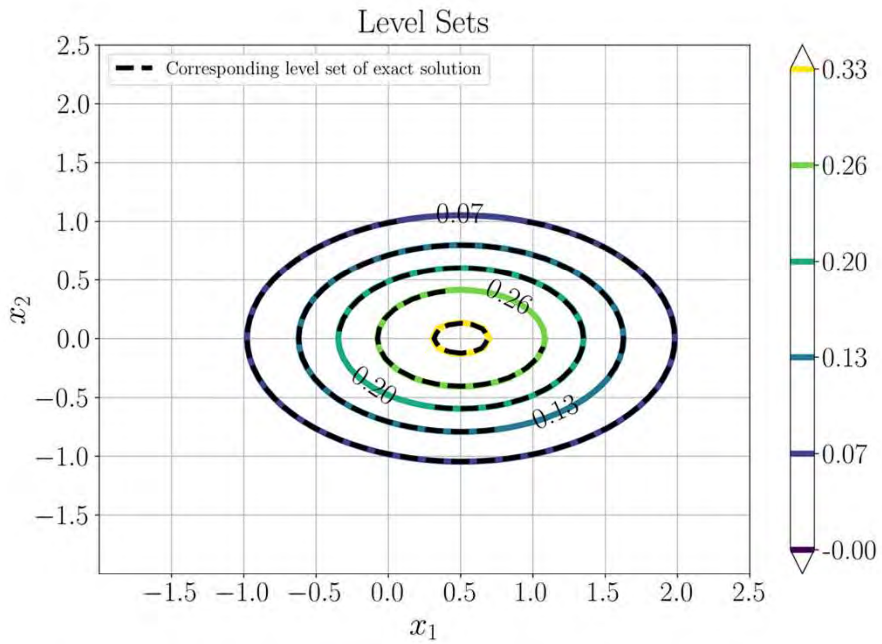



**Figure 2:** Pdf level sets at steady state for $\tau_{\text{cor}} = 1$

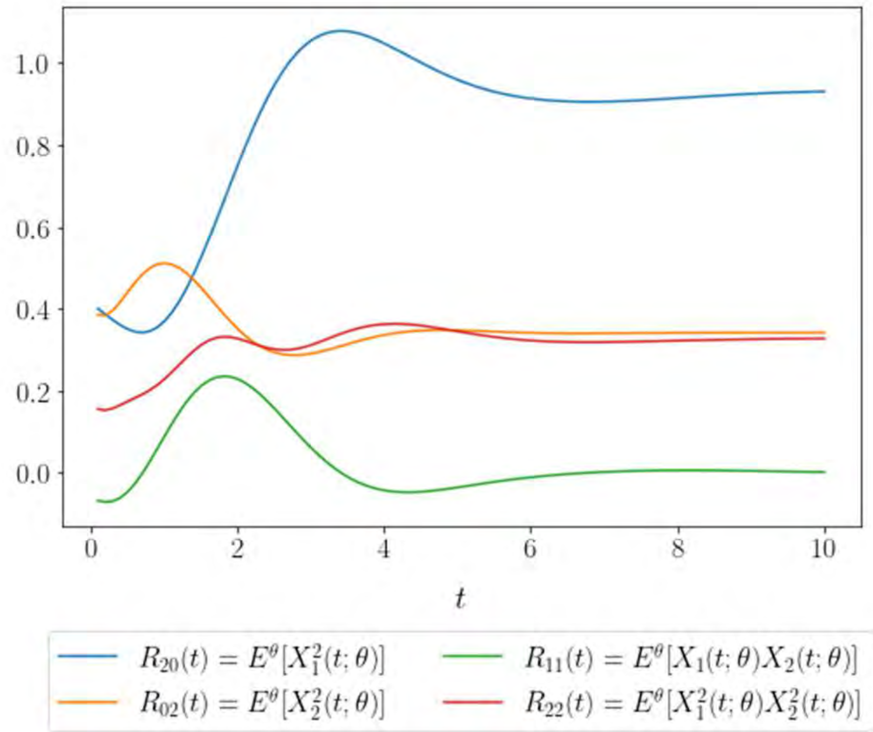



**Figure 3:** evolution of moments

# The Duffing oscillator

The equation of the Duffing oscillator reads

$$m\ddot{X}(t;\theta) + b\dot{X}(t;\theta) + \eta_1 X(t;\theta) + \eta_3 X^3(t;\theta) = \Xi(t;\theta)$$

$$\dot{X}(t_0;\theta) = \dot{X}_0(\theta), \quad X(t_0;\theta) = X_0(\theta),$$

We study the bistable case for

$$m = 1, \quad b = 0.5, \quad \eta_1 = -1 \quad \text{and} \quad \eta_3 = 1.1.$$

- The excitation $\Xi(t;\theta)$, is considered a zero-mean Ornstein-Uhlembeck (OU) process
- Initial value $X_0(\theta)$ is taken uncorrelated to the excitation.

# FPK equation for the Duffing oscillator

Under the assumption of white noise excitation, with autocorrelation function

$$C^{\mathrm{WN}}_{\Xi(\bullet)\Xi(\bullet)}(t, s) = 2 D_{\mathrm{WN}}\, \delta(t - s),$$

the classical FPK equation, corresponding to the Duffing oscillator reads

$$\partial_t f_{\boldsymbol{X}(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\Big[ h_n(x,t)\, f_{\boldsymbol{X}(t)}(\boldsymbol{x}) \Big] = \frac{\partial^2}{\partial x_2^2} D_{\mathrm{WN}}\, f_{\boldsymbol{X}(t)}(\boldsymbol{x})$$

The **drift coefficients** in the above equation read:

$$h_1(x,t) = x_2, \qquad h_2(x,t) = -\eta_1 x_1 - b x_2 - \eta_3 x_1^3.$$

# Transient solution of FPK

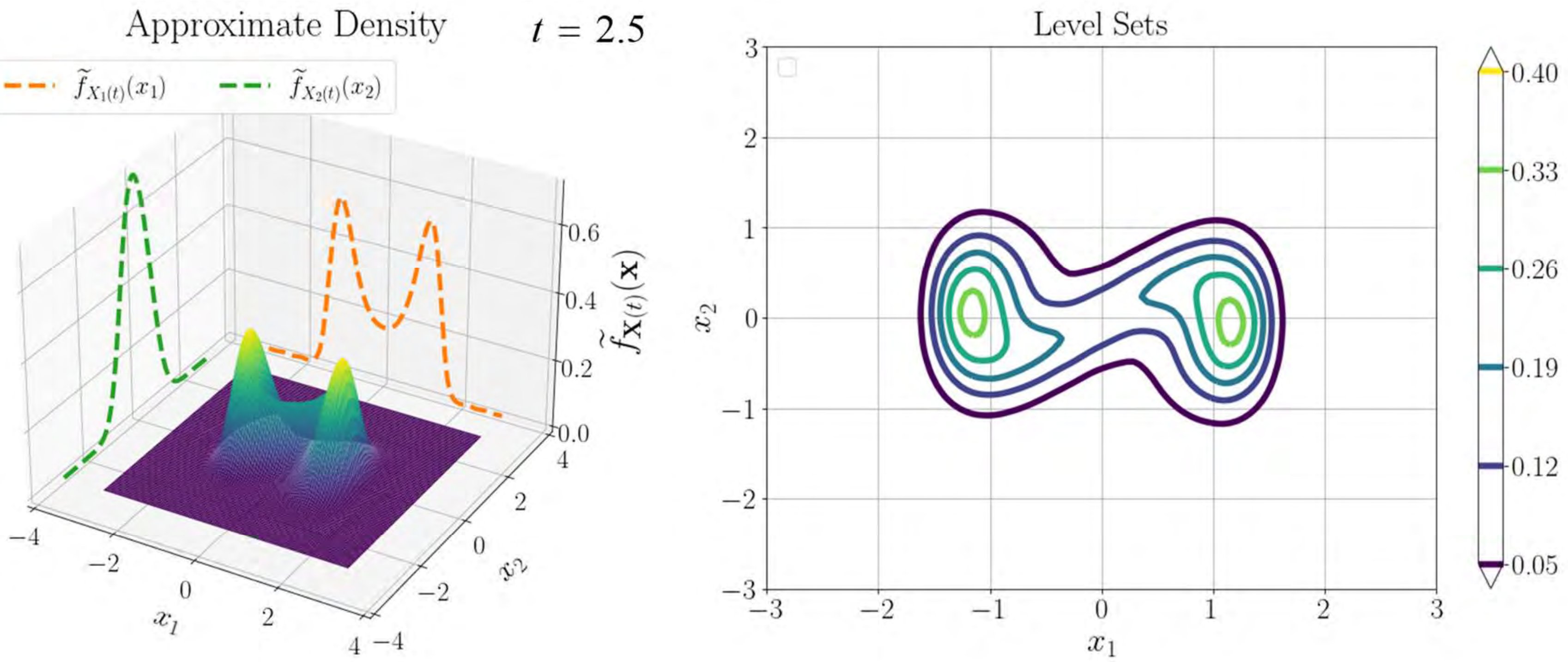



**Figure 4:** PUFEM solution of FPK corresponding to the Duffing oscillator at transient state for $D_{\mathrm{WN}} = 0.12$

# Steady state solution of FPK

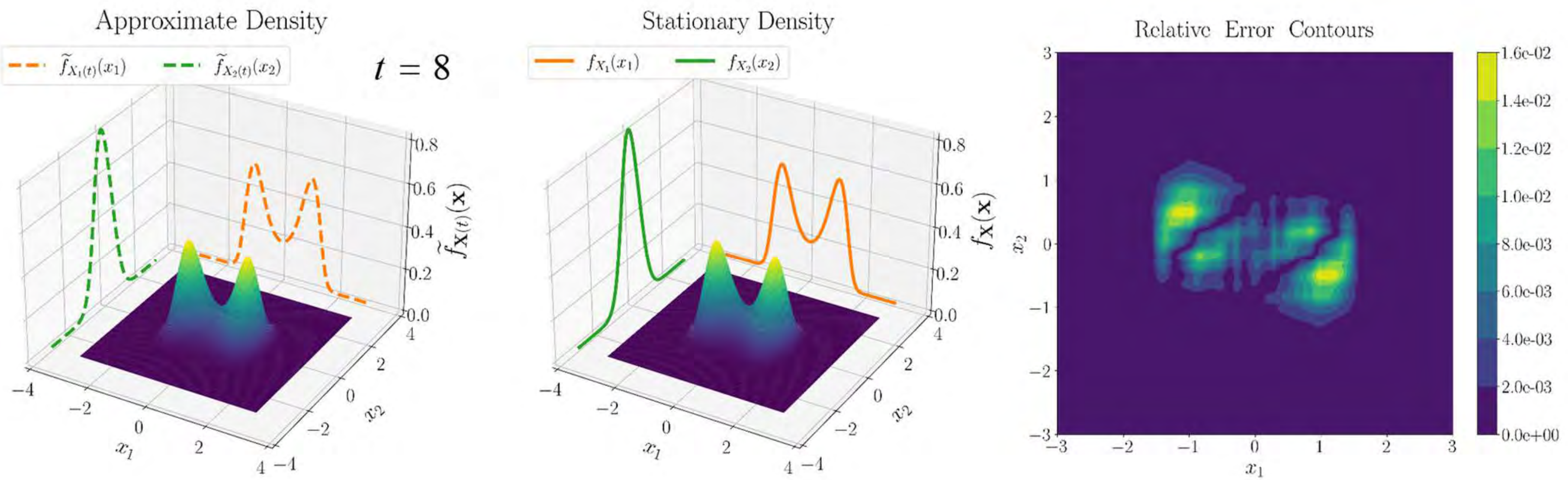



**Figure 5:** PUFEM solution of FPK corresponding to the Duffing oscillator at steady state for $D_{WN} = 0.12$

# Level sets and moments

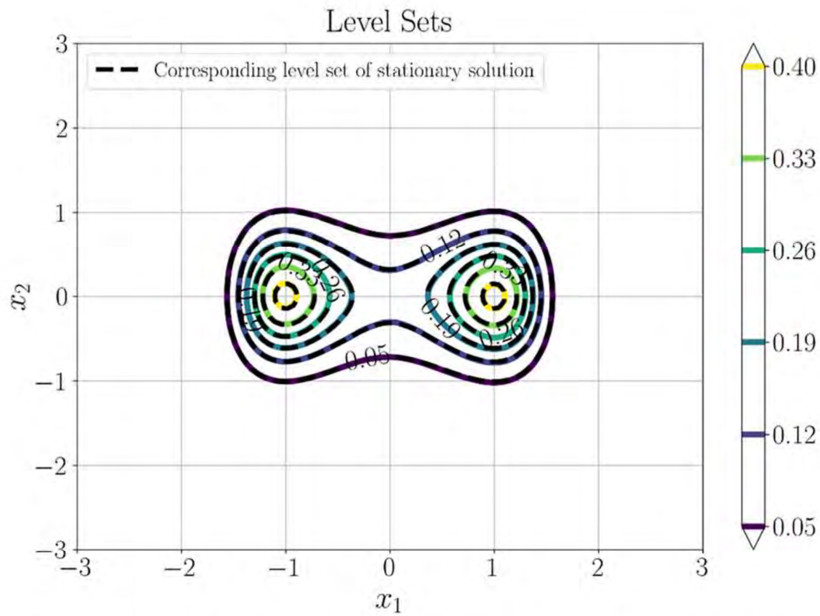



**Figure 6:** Pdf level sets at steady state, $t = 8$. Comparison of numerical and analytic solutions

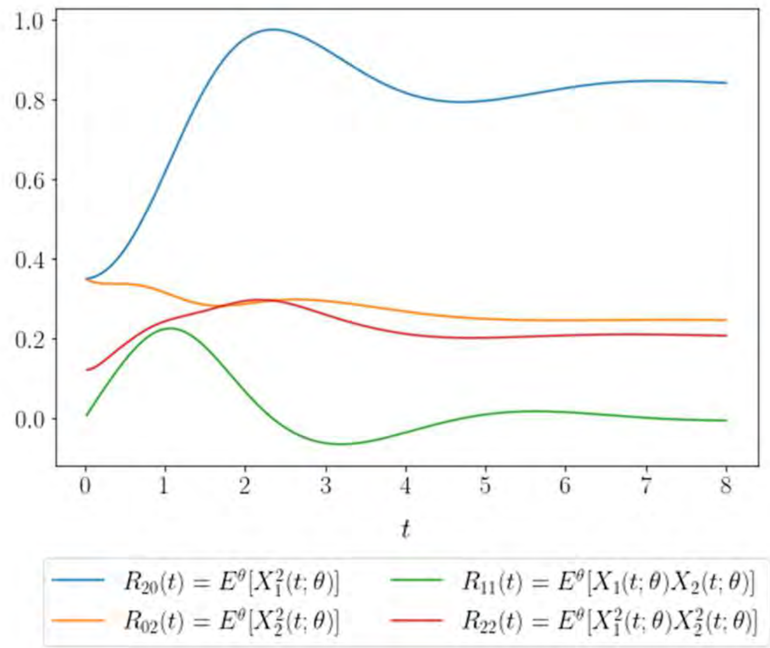



**Figure 7:** Evolution of moments

# Pdf-evolution equation for the Duffing oscillator

The one-time response pdf-evolution equation, corresponding to the oscillator reads

$$\partial_t f_{X(t)}(\boldsymbol{x}) + \sum_{n=1}^{2} \frac{\partial}{\partial x_n}\Big[ h_n(x,t)\, f_{X(t)}(\boldsymbol{x}) \Big] = \sum_{n=1}^{2} \frac{\partial^2}{\partial x_2 \partial x_n}\Big( \mathcal{D}_{2n}\big[ f_{X(\cdot)}(\bullet); \boldsymbol{x}, t \big] \cdot f_{X(t)}(\boldsymbol{x}) \Big).$$

$$f_{X(0)}(\boldsymbol{x}) = f_{X^0}(\boldsymbol{x}).$$

In the above equation, the **drift coefficients** $h_n(x,t)$ reads:

$$h_1(x,t) = x_2, \qquad h_2(x,t) = -\eta_1 x_1 - b x_2 - \eta_3 x_1^3$$

The **diffusion coefficients** are expressed as

$$\mathcal{D}_{2n}^{\Xi(\bullet)\,\Xi(\bullet)}\big[ f_{X(\cdot)}(\bullet); \boldsymbol{x}, t \big] = \int_{t_0}^{t} C_{\Xi_2(\bullet)\Xi_2(\bullet)}(t,s)\, \mathcal{B}_{n2}^{\Xi(\bullet)\,\Xi(\bullet)}\big[ f_{X(\cdot)}(\bullet); \boldsymbol{x}, t, s \big]\, ds.$$

# Evolution of solution

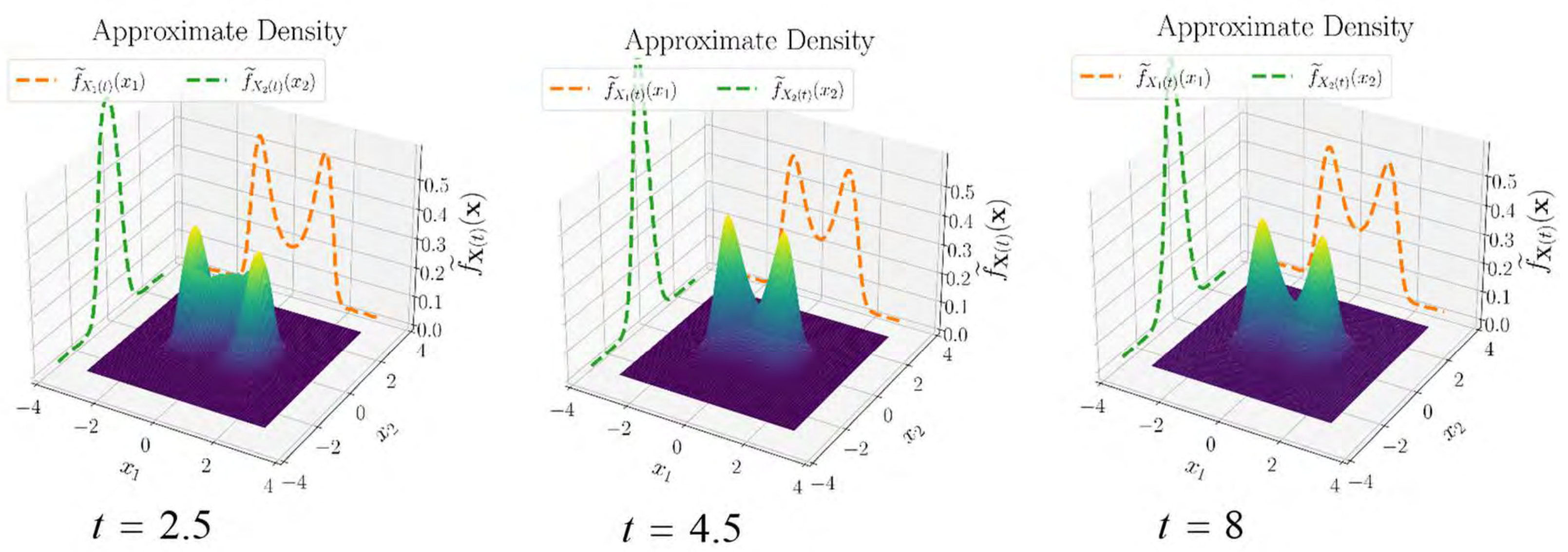



**Figure 8:** PUFEM solution of pdf-evolution equation corresponding to the Duffing oscillator at different times for $\tilde{\tau} = 0.02$

# Evolution of solution

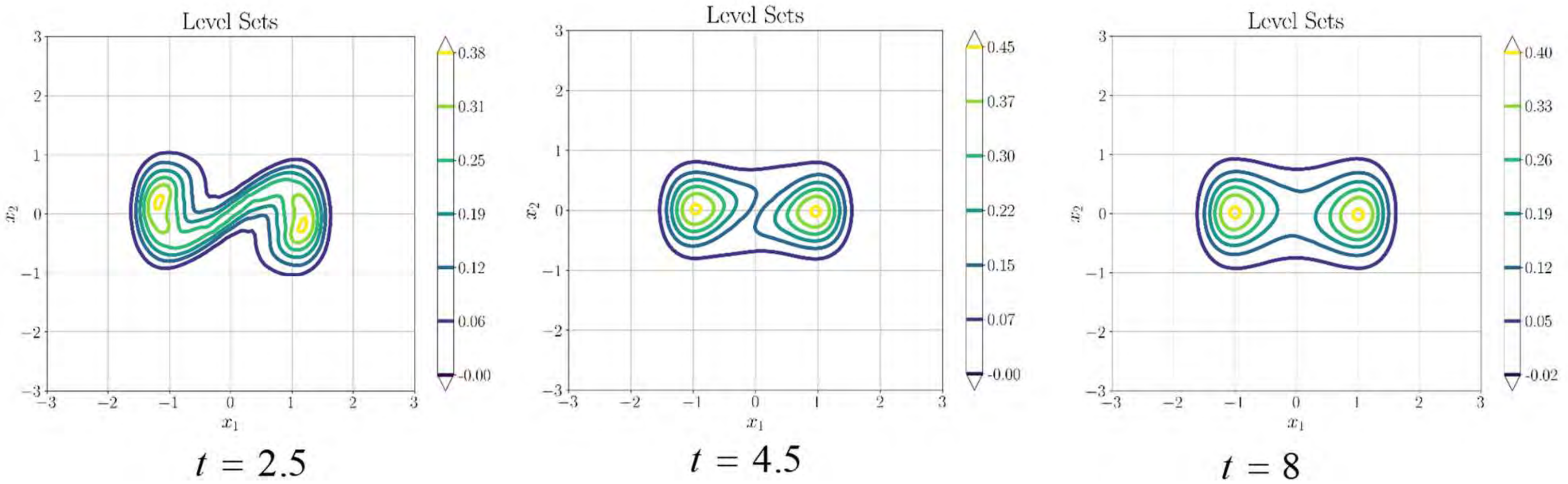



**Figure 9:** Level sets of PUFEM solution at different times, for $\tilde{\tau} = 0.02$

# Evolution of solution

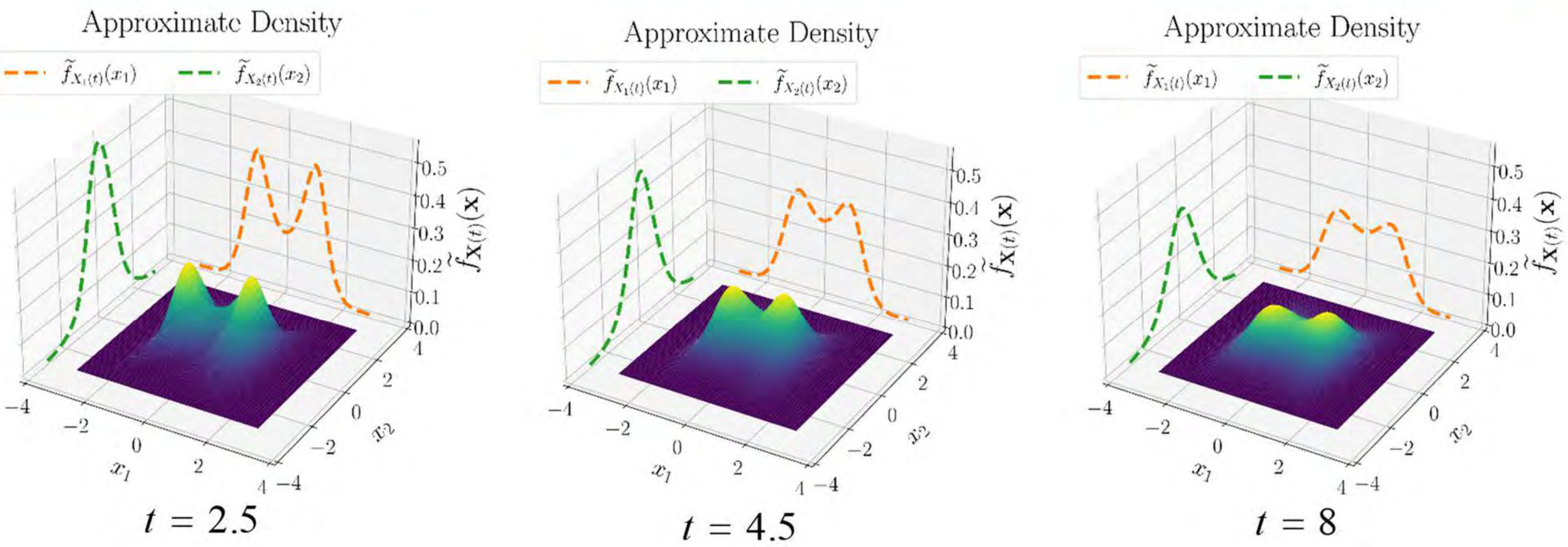



**Figure 10:** PUFEM solution of pdf-evolution equation corresponding to the Duffing oscillator at different times for $\tilde{\tau} = 0.05$

# Evolution of solution

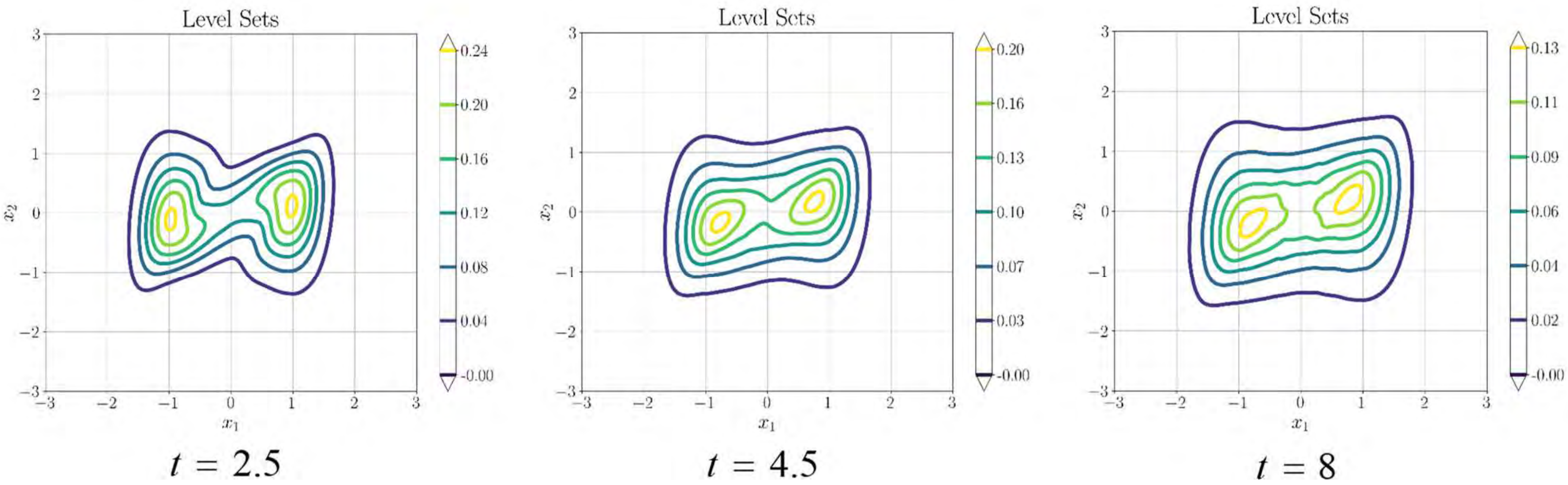



**Figure 11:** Level sets of PUFEM solution at different times, for $\tilde{\tau} = 0.05$

# Evolution of moments

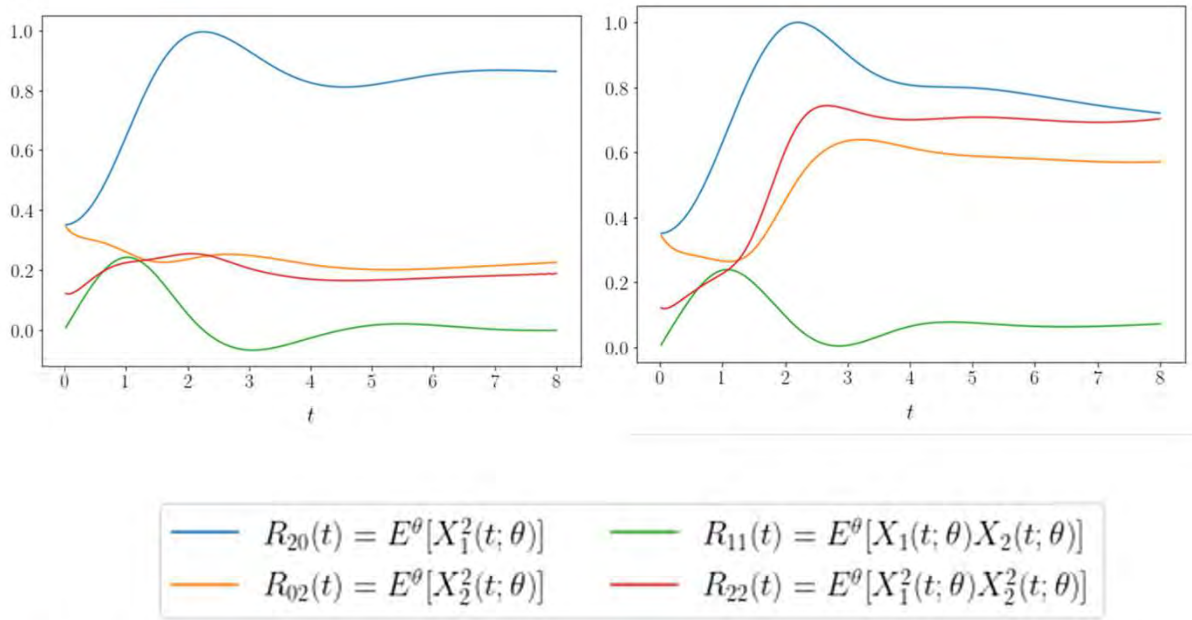



**Figure 12:** Evolution of moments for different correlation times $\tau = 0.02$ (left) and $\tau = 0.05$ (right).

# Future goals

- Verification, via Monte Carlo simulation, of the results concerning the Duffing oscillator

- Investigation and better understanding of the effect of the correlation time and intensity of the excitation

- Study of convergence and error analysis of the partition of unity finite element method and improvement of the numerical scheme .

# References


Babuška, Ivo, Uday Banerjee, and John E Osborn. 2003. 'Survey of Meshless and Generalized Finite Element Methods: A Unified Approach'. *Acta Numerica* 12: 1–125.

Mamis, K. I., G. A. Athanassoulis, and Z. G. Kapelonis. 2019. 'A Systematic Path to Non-Markovian Dynamics: New Response Probability Density Function Evolution Equations under Gaussian Coloured Noise Excitation'. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.*

Mamis, K.I. 2020. 'Probabilistic Responses of Dynamical Systems Subjected to Gaussian Coloured Noise Excitation. Foundations of a Non-Markovian Theory'. National Technical University of Athens.

Melenk, J. M., and I. Babuška. 1996. 'The Partition of Unity Finite Element Method: Basic Theory and Applications'. *Computer Methods in Applied Mechanics and Engineering.*

Oh, Hae-Soo, June G Kim, and Won-Tak Hong. 2008. 'The Piecewise Polynomial Partition of Unity Functions for the Generalized Finite Element Methods'. *Computer Methods in Applied Mechanics and Engineering* 197(45): 3702–11.