From Euler flows with friction to gradient flows

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Introduction

In dynamics of particulate flows/polymers: two widespread theories:

- Smoluchowski theory of diffusion (developed around 1905) that describes motion of particles in a friction dominated regime

$$dx = -\nabla V(x)dt + dB$$

- Kramers and Kirkwood theory (developed between 1940-1950) based on models of Hamiltonian dynamics for many particle systems

$$\begin{aligned} dx &= v\,dt \\ dv &= -\nabla V(x)dt - \tfrac{1}{\varepsilon} v + dW \end{aligned}$$

- The passage from the latter to the former is called Kramers to Smoluchowski approximation.
- High friction or small mass approximation

Euler flows generated by an energy functional

- Hamiltonian Systems driven by an energy $\mathcal{E}(\rho)$

$$\partial_t \rho + \text{div}\,(\rho u) = 0$$
$$\rho \frac{Du}{Dt} = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

  - $\mathcal{E}[\rho]$ is an energy functional, e.g. $\mathcal{E}(\rho) = \int h(\rho) + \kappa(\rho)\,|\nabla(\rho)|^2 dx$

- High friction limit from Hamiltonian flows to gradient flows

$$\partial_t \rho + \text{div}\,(\rho u) = 0$$
$$\varepsilon^2 \rho \frac{Du}{Dt} = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} - \rho u$$

Part I, Euler flows generated by an energy functional

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\rho \frac{Du}{Dt} = \rho\big(\partial_t u + (u \cdot \nabla) u\big) = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

where $\mathcal{E}[\rho]$ is a functional

Hamiltonian

$$\mathcal{H}(\rho, u) = \mathcal{E}(\rho) + \int \tfrac{1}{2} \rho |u|^2 dx$$

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\nabla & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \rho} \\ \frac{\delta \mathcal{H}}{\delta u} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta u} \times \operatorname{curl}_x \big(\frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta u}\big) \end{pmatrix} .$$

$$\frac{d}{dt} \mathcal{H}(\rho, u) = 0$$

ex: the quantum hydrodynamics system

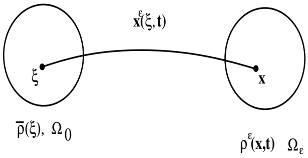
$$
\begin{aligned}
\partial_t \rho + \operatorname{div}(\rho u) &= 0 \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) &= -\nabla p(\rho) + 2\rho \nabla \left(\frac{\triangle \sqrt{\rho}}{\sqrt{\rho}} \right) + \rho \nabla c \\
-\triangle c &= \rho - \bar{\rho}
\end{aligned}
$$

generated by the energy

$$
\mathcal{E}(\rho) = \int h(\rho) + \tfrac{1}{2} \frac{1}{\rho} |\nabla \rho|^2 + \tfrac{1}{2} \rho \, c
$$

with $\rho h''(\rho) = p'(\rho)$

why this structure ?



Family of maps

$$x^\varepsilon(\xi, t) \longrightarrow \begin{cases} u^\varepsilon(x,t) \\ \rho^\varepsilon(x,t) \end{cases}$$

$$\rho^\varepsilon = x^\varepsilon_{\#}\bar\rho\,, \qquad \partial_t \rho^\varepsilon + \operatorname{div}_x(\rho^\varepsilon u^\varepsilon) = 0$$

Find extrema of the action $\mathcal{L}$ over x^ε such that $\rho^\varepsilon(\cdot, t_1) = \rho_1,\ \rho^\varepsilon(\cdot, t_2) = \rho_2$

$$\mathcal{L}[x^\varepsilon] = \int_{t_1}^{t_2} \int_{\Omega_\varepsilon = x^\varepsilon(\Omega_0)} \tfrac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 dxdt - \int_{t_1}^{t_2} \mathcal{E}\left[\rho^\varepsilon(\cdot, t)\right] dt$$

It turns out

$$
\begin{aligned}
x^{\varepsilon}(\xi, t) &= x(\xi, t) + \varepsilon \delta x(\xi, t) \\
\delta x(\xi, t) &= \delta\phi(x(\xi, t), t) \\
\frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} &= -\mathrm{div}_{x}(\rho\delta\varphi)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\int_{t_1}^{t_2} \mathcal{E}\big[\rho^{\varepsilon}(\cdot, t)\big]\,dt\Big) &= \int_{t_1}^{t_2} \big\langle \frac{\delta \mathcal{E}}{\delta\rho}, \frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0}\big\rangle\, d\tau \\
&= \int_{t_1}^{t_2} \big\langle \rho\nabla_x \frac{\delta\mathcal{E}}{\delta\rho}, \delta\varphi\big\rangle\, d\tau
\end{aligned}
$$

Obtain the equations:

$$
\begin{aligned}
\partial_t\rho + \mathrm{div}\,(\rho u) &= 0 \\
\rho\frac{Du}{Dt} &= -\rho\nabla_x\frac{\delta\mathcal{E}}{\delta\rho}
\end{aligned}
$$

$$\mathcal{E}(\rho|\bar\rho) := \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

under hypothesis that $\mathcal{E}(\rho)$ is convex in ρ

Relative energy calculation

$$\begin{aligned} \frac{d}{dt}\left(\int \tfrac{1}{2}\rho|u-\bar u|^2\,dx + \mathcal{E}(\rho|\bar\rho)\right) &= \int -\rho \nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) \\ &\quad + \int \nabla \bar u : S(\rho|\bar\rho)\,dx \end{aligned}$$

where

$$\begin{aligned} \mathcal{E}(\rho|\bar\rho) &:= \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle \\ S(\rho|\bar\rho) &:= S(\rho) - S(\bar\rho) - \left\langle \frac{\delta S}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle \end{aligned}$$

abstract relative energy computation based on

Hypothesis : $\mathcal{E}(\rho)$ satisfies for some functional $S[\rho]$

(*)
$$-\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} = \nabla_x \cdot S[\rho]$$

Formula (*)

- gives meaning to weak solutions
- serves as the basis for the relative energy calculation
- Invariance of $\mathcal{E}(\rho)$ under translations $\rho(\cdot) \to \rho(\cdot + h)$ plus Noether's theorem implies (*)

Application to the quantum hydrodynamics system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\rho \nabla \left(\frac{\triangle \sqrt{\rho}}{\sqrt{\rho}} \right)$$

Thm If (ρ, u) is a weak conservative solution and $(\bar\rho, \bar u)$ smooth conservative soln of QHD then

$$\Psi(t) = \int \tfrac{1}{2}\rho |u - \bar u|^2 + h(\rho|\bar\rho) + \tfrac{1}{2}\rho \left| \frac{\nabla \rho}{\rho} - \frac{\nabla \bar\rho}{\rho} \right|^2 dx$$

satisfies the stability estimate

$$\Psi(t) \le \Psi(0) + O(|\nabla \bar u|) \int_0^T \Psi(\tau) d\tau$$

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Part II, From Euler flows to gradient flows

- Euler flow with high-friction (small-mass approx form)

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\frac{1}{\varepsilon^2}\rho\Big(u + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}\Big)$$

energy dissipation

$$\partial_t\Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2}\rho|u|^2 dx\Big) + \int \rho|u|^2 dx = 0$$

- $\varepsilon \to 0$ limit Diffusion theory

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

$\mathcal{E}[\rho]$ is a convex functional

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big|\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}\Big|^2 dx = 0 \qquad \text{energy dissipation}$$

Let (ρ, u) be an entropy weak solution and $(\bar\rho, \bar u)$ a strong conservative solution of the Euler relaxation system

$$\frac{d}{dt}\left(\mathcal{E}(\rho\,|\bar\rho) + \int \frac{\varepsilon^2}{2}|u-\bar u|^2 dx\right) + \int \rho|u-\bar u|^2 dx$$
$$= -\int \left(\varepsilon^2 \rho \nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) + \nabla \bar u : S(\rho\,|\bar\rho)\right) dx$$

used to compare (ρ, u) and $(\bar\rho, \bar u)$ and to establish convergence results from relaxation system to diffusion theory

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example Bipolar Euler-Poisson model, two electrically charged fluids

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) &= -\rho \nabla \phi - \frac{1}{\tau} \rho u \\
n_t + \nabla \cdot (n v) &= 0 \\
(n v)_t + \nabla \cdot (n v \otimes v) + \nabla p_2(n) &= n \nabla \phi - \frac{1}{\tau} n v \\
-\Delta \phi &= \rho - n
\end{aligned}
$$

Energy identity

$$
\frac{d}{dt} \left(\int \tfrac{1}{2} \rho |u|^2 + \tfrac{1}{2} n |v|^2 \, dx + \mathcal{E}(\rho, n) \right) + \frac{1}{\tau} \int \rho |u|^2 + n |v|^2 \, dx
$$

$$
\begin{aligned}
\mathcal{E}(\rho, n) &= \int_\Omega h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2 dx, \\
-\Delta \phi &= \rho - n
\end{aligned}
$$

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Convergence as $\tau \to 0$ to the bipolar drift-diffusion model used in the analysis of semiconductors (after scaling $t \to \frac{t}{\tau}$, $u \to \tau u$, $v \to \tau v$)

$$
\begin{cases}
\rho_t = \nabla \cdot \big(\nabla p_1(\rho) + \rho \nabla \phi \big) \\
n_t = \nabla \cdot \big(\nabla p_2(n) - n \nabla \phi \big) \\
-\Delta \phi = \rho - n.
\end{cases}
$$

Part III, Diffusion as Gradient Flow in Wasserstein

- $$\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

  Examples: porous media, generalized Keller-Segel models, Cahn-Hilliard equation fit under this framework for various choices of $\mathcal{E}[\rho]$

  Otto, Carillo-Toscani, Villani, Westdickenberg, Ambrosio-Gigli-Savare ...

- Diffusion theory arises by variational minimization based on Wasserstein distance, Jordan-Kinderlehrer-Otto scheme

  ρ^{n+1} is the minimizer of the problem $\quad \min \left\{ \frac{1}{2\tau} d_W(\rho, \rho^n)^2 + \mathcal{E}[\rho] \right\}$

- Brenier-Benamou formula

$$d_W(\rho_0, \rho_1)^2 = \inf_{(\rho,u)} \left\{ \tau \int_0^\tau \int \rho |u|^2 \, dxdt \;\middle|\; \begin{array}{c} \partial_t \rho + \mathrm{div}\, \rho u = 0 \\ \rho(0) = \rho_0 \,,\ \rho(\tau) = \rho_1 \end{array} \right\}$$

$$\partial_t \rho = \text{div}\Big(A(x) \nabla \rho \Big) \qquad\qquad A(x) > 0 \text{ and symmetric}$$

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$$= \operatorname{div}\Big(\rho A(x) \nabla \frac{\delta \mathcal{E}}{\delta \rho} \Big) \qquad \mathcal{E}(\rho) = \int \rho \ln \rho \, dx$$

Visualize this diffusion as

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -A(x) \nabla \frac{\delta \mathcal{E}}{\delta \rho}$$

small mass approximation of the Euler system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\varepsilon^2 \rho \Big(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \Big) = -\Big(\rho B(x) u + \rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} \Big) \qquad B(x) = A^{-1}(x) > 0$$

$$\partial_t \Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2} \rho |u|^2 dx \Big) + \int \rho u \cdot B(x) u dx = 0$$

Analog of the Brenier-Benamou formula

$$W_A(\rho_0,\rho_1)^2 = \inf_{(\rho,v)} \left\{ \tau \int_0^\tau \int v \cdot B(x) v \ \rho dx ds \ \Big| \ \begin{array}{r} \partial_s \rho + \mathrm{div}\, \rho v = 0 \\ \rho(0) = \rho_0 \, , \ \rho(\tau) = \rho_1 \end{array} \right\}$$

- minimum is achieved
- $B(x) = (\nabla_x b)^T (\nabla_x b)$ $\qquad b : (\mathbb{R}^d, B) \to (R^N, Euclidean)$
  secured by isometric embedding theorem of Nash-Kuiper
- defines a 2-Wasserstein distance associated to the friction matrix $A(x)$ (or the mobility matrix $B(x)$)

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Analog of the Jordan-Kinderlehrer-Otto scheme

$$\rho^{n+1} \text{ is the minimizer of the problem} \quad \min_{\rho \in K} \left\{ \frac{1}{2\tau} W_A(\rho, \rho^n)^2 + \int \rho \ln \rho \, dx \right\}$$

Variational scheme approximates implicit Euler Scheme of the form

$$\frac{\rho^{n+1} - \rho^n}{\tau} = \operatorname{div}_x \left(\rho^{n+1} A(x) \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} (\rho^{n+1}) \right)$$

and as $\tau \to 0$, $\rho^\tau(x,t) \to \rho(x,t)$ with

$$\partial_t \rho = \operatorname{div} \left(\rho A(x) \nabla \ln \rho \right)$$

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the STALKER of our youth

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STALKER a film by ANDREI TARKOVSKY

A guide leads two men through an area known as the Zone to find a room that grants wishes.