Optimal management of stochastic shallow lakes

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oligotrophic vs eutrophic lakes

Modelling the nutrient content

The nutrient content is usually measured in terms of P concentration.

$$\begin{aligned} \dot{P}(t) = L(t) \qquad & \text{(P loading by natural and human activity)} \\ - sP(t) \qquad & \text{(sedimentation, outflow)} \\ + \Phi\big(P(t)\big) \qquad & \text{(recycling from sediments)} \end{aligned}$$

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Limnologists take Φ to be a sigmoid function, typically

$$\Phi(x) = r\,\frac{x^2}{m^2 + x^2}.$$

[Carpenter, Ludwig, Brock 1999]

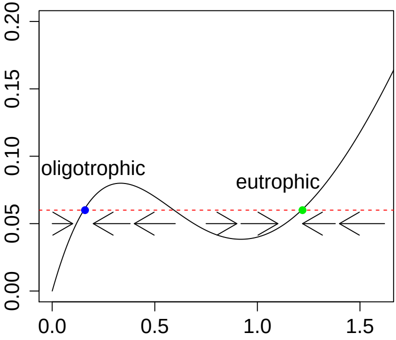
With a change of variables ($x = \frac{P}{m},\; a = \frac{L}{r},\; b = \frac{sm}{r}$) the equation becomes

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}.$$

Equilibrium under constant load

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}.$$

When b is not too large the lake may have 2 stable equilibria.



Solid black line: $y = bx - \frac{x^2}{1+x^2}$. Dashed red line: $y = a$.

A welfare function

Farmers or industry have an interest to increase P loading, a.

Visitors prefer a clean lake, i.e. small x.

Suppose a community balances these needs and assigns value to the state of the lake

$$U(a, x) = \ln a - cx^2.$$

Given the current P concentration x, we are interested in the optimal loading $\{a(t) : t \geq 0\}$ to maximise the welfare function

$$J(x, a(\cdot)) = \int_0^\infty e^{-\rho t}\, U(a(t), x(t))dt$$

where $\{x(t) : t \geq 0\}$ solves

$$\dot{x}(t) = a(t) - bx(t) + \frac{x^2(t)}{1 + x^2(t)}, \qquad x(0) = x.$$

The problem

Add multiplicative noise [Grass, Kiseleva, Wagener 2015]

$$\begin{cases} dx(t) = \left(u(t) - bx(t) + \dfrac{x^2(t)}{x^2(t)+1} \right) dt + \sigma x(t) dW(t), \\ x(0) = x \end{cases} \tag{1}$$

and the value function

$$V(x) = \sup_{u \in \mathfrak{U}_x} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left[\ln u(t) - cx^2(t) \right] dt \right].$$

Admissible controls $u \in \mathfrak{U}_x$ should be positive, adapted processes in some filtered probability space such that

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} \ln u(t) dt \right] < \infty$$

and (1) has a unique strong solution.

DPP and HJB equation

The tool to characterise the value function V is the Dynamic Programming Principle (DPP):

$$V(x) = \sup_{u \in \mathfrak{U}_x} \mathbb{E}\left[\int_0^{\theta_u} e^{-\rho t}\big(\ln u(t) - cx^2(t)\big)dt + e^{-\rho\theta_u} V(x(\theta_u))\right].$$

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- Measurable selection problems in the proof of DPP
- No a priori regularity for V. In fact we do not even know if V takes finite values.
- Unbounded controls, the Hamiltonian may be infinite.
- Boundary conditions at zero? at infinity?

V as a constrained HJB v.s.

[Kossioris, L, Souganidis 2016]

The value function V is a continuous constrained viscosity solution on $[0, \infty)$ to the HJB equation

$$\rho V = \underbrace{\left[\left(\frac{x^2}{x^2+1} - bx \right) V' - \left(\ln(-V') + cx^2 + 1 \right) + \frac{1}{2}\sigma^2 x^2 V'' \right]}_{H(x,V',V'')}.$$

i) For every $\phi \in C^2[0, \infty)$ such that $V - \phi$ has a local maximum at $x \geq 0$:

$$\rho V(x) \leq H\left(x, \phi'(x), \phi''(x)\right)$$

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i) For every $\phi \in C^2[0,\infty)$ such that $V-\phi$ has a local maximum at $x \geq 0$:

$$\rho V(x) \leq H\big(x, \phi'(x), \phi''(x)\big)$$

ii) For any $\phi \in C^2(0,\infty)$ such that $V-\phi$ has a local minimum at $x > 0$:

$$\rho V(x) \geq H\big(x, \phi'(x), \phi''(x)\big)$$

V as a constrained HJB v.s.

[Kossioris, L, Souganidis 2016; Koutsimpela, L 2022+]

The value function V is the unique continuous constrained viscosity solution on $[0,\infty)$ to

$$\rho V - \underbrace{\left[\left(\frac{x^2}{x^2+1} - bx\right)V' - \left(\ln(-V') + cx^2 + 1\right) + \frac{1}{2}\sigma^2 x^2 V''\right]}_{H(x,V',V'')} = 0.$$

satisfying:

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satisfying:

$$DV(x) \leq -\frac{1}{c_*} < 0, \quad \forall x \in [0, \infty).$$

and

$$\liminf_{x \to \infty} \frac{V(x)}{1+x^2} > -\infty.$$

Properties of the value function $(\sigma^2 < 2b + \rho)$

- V decreases and $0 \leq x < y \Longrightarrow \frac{V(y)-V(x)}{y-x} \leq -c < 0,$

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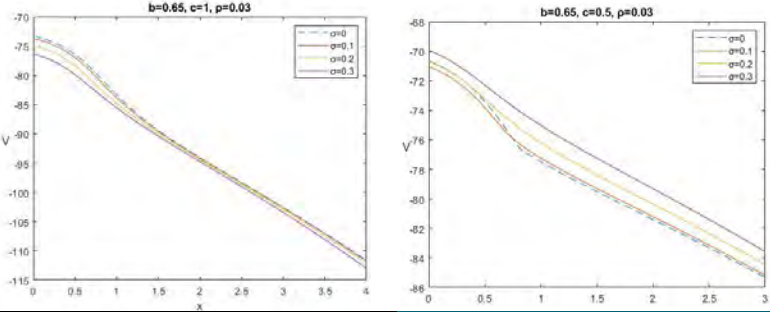
- For $\sigma > 0$, V is smooth and satisfies the HBJ classically. For $\sigma = 0$, V' may have a jump discontinuity at one $x_* > 0$
- V is differentiable at 0 and $\ln\big(-V'(0)\big) + \rho V(0) + 1 = 0.$
- $$V(x) + A\left(x + \frac{1}{b+\rho}\right)^2 + \frac{1}{\rho}\ln\left(x + \frac{1}{b+\rho}\right) \overset{x\to\infty}{\longrightarrow} K$$



Optimally controlled process

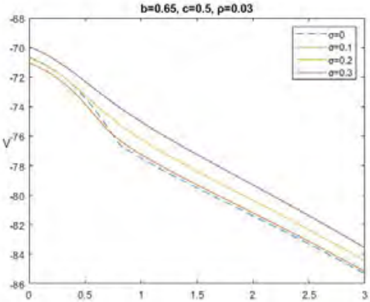
Bartaloni '20,'21; Koutsimpela, L 2022+

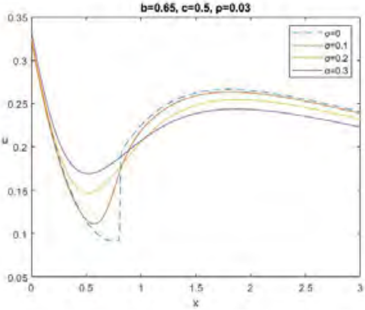
A verification theorem gives the optimal control in feedback form

$$u_*(x(t)) = -\frac{1}{V'_\sigma(x(t))} \leq \frac{1}{c_*}$$

so the optimally controlled system satisfies

$$dx(t) = \left(-\frac{1}{V'_\sigma(x(t))} - bx(t) + \frac{x^2(t)}{x^2(t)+1}\right) dt + \sigma x(t) dW(t).$$





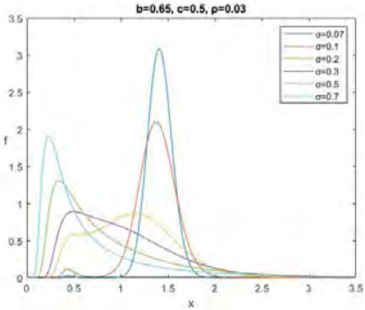
Invariant measure

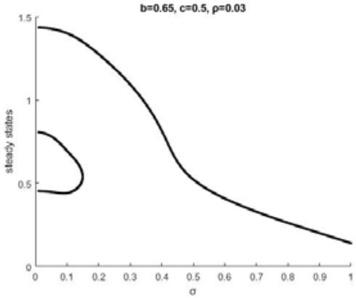
$$\mathcal{L}^* \mu = 0 \Longrightarrow d\mu(x) = \frac{1}{Z}\, x^{-2\left(1+\frac{b}{\sigma^2}\right)}\, e^{-\Psi_\sigma(x)}\, dx.$$

The exponent Ψ_σ is explicitly given in terms of V'_σ and

$$\Psi_\sigma(x) \simeq \frac{2}{\sigma^2 |V'_\sigma(0)| x}, \;\; x \to 0 \qquad \text{and} \qquad \Psi_\sigma(x) \simeq \frac{2}{\sigma^2 x}, \;\; x \to \infty.$$

Polynomial tails at infinity get fatter as σ increases.

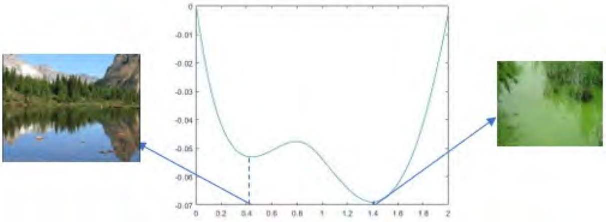




Oligotrophic vs Eutrophic

When σ is small and other parameters are suitable, the invariant distribution may be bimodal. The process $y(t) = \ln\big(x(t)\big)$ is a diffusion in a double-well potential $\Phi_\sigma(y)$:

$$dy(t) = -\Phi'_\sigma\big(y(t)\big)dt + \sigma\, dW(t).$$



with invariant distribution for $\sigma > 0$

$$d\mu_\sigma(x) = \frac{1}{Z_\sigma}\exp\Big(-\frac{2}{\sigma^2}\Phi_\sigma(x)\Big)dx.$$

Deterministic vs Stochastic trajectories

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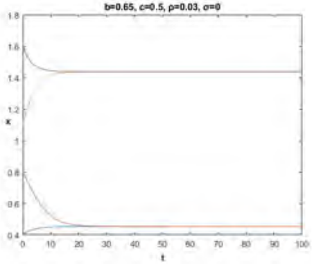
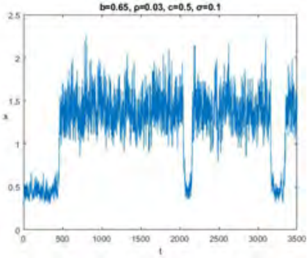




$\sigma \to 0$ asymptotics: metastability

Koutsimpela, L. 2022+

V_σ semi-convex $\Rightarrow V_\sigma' \to V_0'$, as $\sigma \to 0$, $\forall x \neq x_*$

$\Rightarrow \Phi_\sigma \to \Phi_0$, uniformly on compact subsets of $(0, +\infty)$

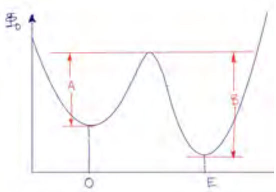
reduced to Freidlin-Wentzel theory.

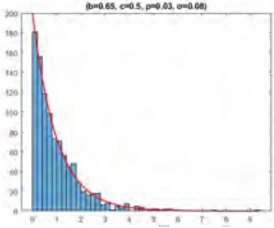
Arrhenius law : $\frac{\sigma^2}{2} \log \mathbb{E}\left[\tau_{O \to E}\right] \to A, \quad \frac{\sigma^2}{2} \log \mathbb{E}\left[\tau_{E \to O}\right] \to B$

[Siguira 1993, Bovier & den Hollander book 2014]

$$\frac{\tau_{O \to E}}{\mathbb{E}\left[\tau_{O \to E}\right]} \xrightarrow{d} \text{Exp}(1)$$

[Day 1983]





Ευχαριστούμε Μάκη.

Πάντα να εμπνέεις τους νεότερους!